A Two-Step Proximal Point Algorithm for Nonconvex Equilibrium Problems with Applications to Fractional Programming

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Abstract

We present a proximal point type algorithm tailored for tackling pseudomonotone equilibrium problems in a Hilbert space which are not necessarily convex in the second argument of the involved bifunction. Motivated by the extragradient algorithm, we propose a two-step method and we prove that the generated sequence converges strongly to a solution of the nonconvex equilibrium problem under mild assumptions and, also, we establish a linear convergent rate for the iterates. Furthermore, we identify a new class of functions that meet our assumptions, and we provide sufficient conditions for quadratic fractional functions to exhibit strong quasiconvexity. Finally, we perform numerical experiments comparing our algorithm against two alternative methods for classes of nonconvex mixed variational inequalities.

Keywords: Proximal point methods; Nonconvex optimization; Equilibrium problems; Generalized convexity; Fractional programming.

1 Introduction

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Introduced in the pioneering work of Ky Fan [32], and studied deeply in the last fifty years (see [6, 9, 12, 25] among others), equilibrium problems are considered a very general formulation since they encompass several mathematical

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problems found in fixed point theory, optimization and nonlinear analysis, like minimization problems, linear complementary problems, variational inequalities and multiobjective optimization problems among others.

In the last decades, equilibrium problems have been deeply studied in the literature from both theoretical and algorithmic viewpoints, specially in the convex framework. In the nonconvex case, existence results for pseudomonotone equilibrium problems in the general quasiconvex case were obtained in [12, 20, 22, 34].

Proximal point algorithms (PPA henceforth), introduced in [36, 37, 46], were intensively study since then (see for instance [4]) for dealing with maximal monotone operators and convex problems. They have been extended to equilibrium problems which are convex in the second argument of the bifunction (see [9, 10, 29, 30, 40, 45] among others). In general, these algorithms assume monotonicity of the bifunction. This monotonicity assumption has been relaxed (e.g., to pseudomonotonicity), but the only instance in which the assumption of convexity of the bifunction in its second argument is weakened, is, as far as we know, the method in [24]. This reference presents a PPA for solving pseudomonotone equilibrium problems with quasiconvex (and possibly nonconvex) bifunctions. Following the research line of [24], and considering also the two-step method for pseudomonotone equilibrium problems presented in [5, page 54] for the convex case, we introduce in this paper a two-step PPA for dealing with pseudomonotone equilibrium problems with strongly quasiconvex bifunctions.

Some comments are due on two-step algorithms in general. They have been introduced in several settings, with the goal of improving the convergence properties of their one-step versions. We can mention the well known predictor-corrector methods for solving ordinary differential equations, and, more in vein with our case, the extragradient method for finding zeros of monotone operators, presented for the first time in [31], which we describe next. The natural extension of the gradient method for optimization to the context of finding a zero of an operator $T : \mathcal{H} \to \mathcal{H}$, consists of considering an iteration of the form $x^{k+1} = x^k - \alpha_k T(x^k)$, with a positive stepsize α_k. A suitable choice of the stepsize guarantees convergence to a zero when T is strongly monotone and such a zero exists, but simple examples with operators which are just monotone (e.g., rotations in dimension 2) exhibit divergence for any choice of the stepsize. This obstacle is circumvented through the introduction of a second step in each iteration, namely getting first an auxiliary point $y^k = x^k - \alpha_k T(x^k)$, and then obtaining the next iterate as $x^{k+1} = x^k - \beta_k T(y^k)$, where β_k is another positive stepsize. With a suitable choice of the stepsizes, the algorithm turns out to be convergent for all monotone operators, as long as they have a zero.

In this paper, our method has a basic step similar to the iteration in [24], yet it works in the realm of Hilbert spaces and adopts a two-step approach, incorporating an auxiliary step akin to the extragradient method. It allows us to deal with the nonconvex case, because it requires strong quasiconvexity in the second argument of the bifunction, instead of convexity, and leads to convergence properties better than those in [24], like, for instance, a linear convergence rate, with an asymptotic error constant explicitly determined in terms of the parameters of the method.

Furthermore, we extend the utility of our method by applying it to fractional programming problems and inverse mixed variational inequalities (IMVIs), both of which give rise to equilibrium problems featuring bifunctions strongly quasiconvex in their second argument, yet not convex. In Example 4.1, we introduce a broad array of IMVIs falling within this category.

We conclude by presenting numerical experiments comparing our algorithm with a one-step PPA and an Extragradient method, both appropriate for solving nonconvex pseudomonotone equilibrium problems, introduced in [24] and [57], respectively. Our algorithm proves to be clearly superior to the Extragradient one, and exhibits a more stable behavior than the one-step proximal method.

We make some comments now on the relevance of considering equilibrium problems with quasiconvex bifunctions. Firstly, considering equilibrium problems with quasiconvex bifunctions and striving for strong convergence in the Hilbert setting is of utmost relevance. Quasiconvexity is a very natural extension of convexity: while convex functions are those with a convex epigraph, quasiconvex functions are those with convex sublevel sets.

Moreover, many interesting applications deal with quasiconvex functions, which are not convex. For instance, quasiconvex functions appear in Economics and Financial Theory, especially in consumer preference theory (see [8, 38, 54]), since quasiconcavity is the mathematical formulation of the natural assumption of a *tendency to diversification*. Such functions are quasiconcave under natural assumptions on consumer preferences, while preferences that result in concave utility functions are often considered artificial (see, for instance, [8, 38]). Another significant case arises in fractional min-max programming, where the objective function is the maximum of quotients with nonnegative convex numerators and positive concave denominators. These functions are quasiconvex but not convex in general.

In the realm of numerical algorithms for Hilbert spaces, achieving strong convergence (as opposed to weak) is certainly relevant. While numerous techniques have been developed towards this end (see, for instance, [41, 48, 52, 55]), our proposed algorithm stands out by attaining strong convergence without relying on strategies employed in these cited works. Instead, it capitalizes solely on the inherent strong quasiconvexity structure of the bifunction and the judicious choice of the regularized parameters.

The scheme of the paper is as follows: Section 2 contains preliminary results, Section 3 introduces the algorithm and develops its convergence analysis, including the proof of linear convergence, and Section 4 presents examples and numerical experiments.

2 Preliminaries and Basic Definitions

sec:2

We denote $\mathcal{H}$ as a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$. By "$\rightharpoonup$" we mean weak convergence while "$\to$" means strong convergence, thus the weak limit of a sequence $\{x^k\}_k$ in $\mathcal{H}$ (whenever it exists) will be denoted by $w\text{-}\lim_{k\to+\infty} x^k$ while the strong limit by $\lim_{k\to+\infty} x^k$. The set $]0,+\infty[$ is denoted by $\mathbb{R}_{++}$ and, given a matrix $A \in \mathbb{R}^{n\times n}$, its smallest eigenvalue is denoted by $\lambda_{\min}(A)$.

Given any $x, y, z \in \mathcal{H}$ and any $\beta \in \mathbb{R}$, the following relation holds:

$$\langle x - z, y - x\rangle = \frac{1}{2}\|z - y\|^2 - \frac{1}{2}\|x - z\|^2 - \frac{1}{2}\|y - x\|^2. \tag{2.1}$$

3:points

Given a convex and closed set K, we denote the projection of u on K by $P_K(u)$. A well-known property of P_K is the following:

$$v = P_K(u) \iff \langle u - v, w - v\rangle \le 0,\ \forall\, w \in K. \tag{2.2}$$

proj_K

Let $h : \mathcal{H} \to \overline{\mathbb{R}}$ be an extended-valued function. Its effective domain is defined by $\operatorname{dom} h := \{x \in \mathcal{H} : h(x) < +\infty\}$, its epigraph by $\operatorname{epi} h := \{(x,t) \in \mathcal{H}\times\mathbb{R} : h(x) \le t\}$, its sublevel set at $\lambda \in \mathbb{R}$ by $S_\lambda(h) := \{x \in \mathcal{H} : h(x) \le \lambda\}$ and its set of minimizers by $\operatorname{argmin}_{\mathcal{H}} h$. It is said that h is a proper function when $\operatorname{dom} h$ is nonempty and $h(x) > -\infty$ for all $x \in \mathcal{H}$ and that h is lower semicontinuous (lsc henceforth) at $\overline{x} \in \mathcal{H}$ if for any sequence $\{x_k\}_k \subseteq \mathcal{H}$ with $x_k \to \overline{x}$, $h(\overline{x}) \le \liminf_{k\to+\infty} h(x_k)$.

A function h with convex domain is said to be

(a) convex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y),\ \forall\, \lambda \in [0,1], \tag{2.3}$$

def:convex

(b) strongly convex on $\operatorname{dom} h$ with modulus $\gamma \in]0,+\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \le \lambda h(y) + (1-\lambda)h(x) - \lambda(1-\lambda)\frac{\gamma}{2}\|x - y\|^2, \tag{2.4}$$

strong:convex

(c) semistrictly quasiconvex if, given any $x, y \in \operatorname{dom} h$, with $h(x) \neq h(y)$, then

$$h(\lambda x + (1-\lambda)y) < \max\{h(x), h(y)\},\ \forall\, \lambda \in]0,1[, \tag{2.5}$$

(d) quasiconvex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1-\lambda)y) \le \max\{h(x), h(y)\},\ \forall\, \lambda \in [0,1], \tag{2.6}$$

def:qcx

(e) strongly quasiconvex on $\operatorname{dom} h$ with modulus $\gamma \in]0,+\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \le \max\{h(y), h(x)\} - \lambda(1-\lambda)\frac{\gamma}{2}\|x - y\|^2. \tag{2.7}$$

strong:quasiconvex

It is said that h is strictly convex (resp. strictly quasiconvex) if the inequality in (2.3) (resp. (2.6)) is strict whenever $x \neq y$.

The relationship between all these notions is summarized below (we denote quasiconvex by qcx):

$$\begin{array}{ccccccc}
\text{strongly convex} & \Longrightarrow & \text{strictly convex} & \Longrightarrow & \text{convex} & \Longrightarrow & \text{qcx} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{strongly qcx} & \Longrightarrow & \text{strictly qcx} & \Longrightarrow & \text{semistrictly qcx} & & \\
 & & \Downarrow & & & & \\
 & & \text{qcx} & & & &
\end{array}$$

If, in addition, the function is lsc, then functions in all previous classes are quasiconvex. However, the reverse statements do not hold in general. For instance, the Euclidean norm $h_1(x) = \|x\|$ is strongly quasiconvex without being strongly convex on any bounded convex set (see [27, Theorem 2]) and the function $h_2(x) = \frac{x}{1+|x|}$ is strictly quasiconvex without being strongly quasiconvex on $\mathbb{R}$ while the others are well-known (see [7, 16]).

Recall that a proper function $h : \mathcal{H} \to \overline{\mathbb{R}}$ is said to be:

(i) 2-supercoercive, if
$$\liminf_{\|x\| \to +\infty} \frac{h(x)}{\|x\|^2} > 0, \tag{2.8}$$

(ii) coercive, if
$$\lim_{\|x\| \to +\infty} h(x) = +\infty. \tag{2.9}$$

or equivalently, if $S_\lambda(h)$ is bounded for all $\lambda \in \mathbb{R}$.

Clearly, every 2-supercoercive function is coercive, but the converse statement does not hold as the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \frac{|x|}{1+|x|}$ shows.

In [33, Theorem 1], the author proved that every strongly quasiconvex is 2-supercoercive in the finite dimensional case. In Hilbert spaces, we were able to ensure coerciveness under an additional assumption (similar to the one proposed in [2]) as we show next.

exist:unique **Proposition 2.1.** *Let $K \subseteq \mathcal{H}$ be a convex set and $h : \mathcal{H} \to \overline{\mathbb{R}}$ be a proper, lsc and strongly quasiconvex function on $K \subseteq \operatorname{dom} h$ with modulus $\gamma > 0$. Assume in addition that:*

$(P0)$ *If $\{t_n\}_n \subset \mathbb{R}, \{v_n\}_n \subset \mathcal{H}, t_n \to +\infty, v_n \rightharpoonup v$, and $\{h(t_n v_n)\}_k$ is bounded from above, then*
$$\|v_n - v\| \to 0.$$

Then h is coercive.

Proof. Suppose that h is not coercive. Then there exist $\{x_n\}_n \subseteq \mathcal{H}$ and $M > 0$ such that $\|x_n\| \to +\infty$, as $n \to +\infty$, and $h(x_n) < M$ for all $n \in \mathbb{N}$. We may assume that $\frac{x_n}{\|x_n\|} \rightharpoonup u$. For every $n \in \mathbb{N}$, taking $t_n := \|x_n\|$ and $v_n := \frac{x_n}{\|x_n\|}$, we obtain $h(t_n v_n) = h(x_n) < M$, thus by $(P0)$ we obtain that $\|v_n - u\| \to 0$, i.e., $\frac{x_n}{\|x_n\|} \to u$ and $\|u\| = 1$.

Let $y_n = \frac{\|x_n\|}{1+\|x_n\|}x_1 + \frac{1}{1+\|x_n\|}x_n$. Then $y_n \to x_1 + u$, and since h is lsc, we have

$$h(x_1+u) \leq \liminf_{n\to+\infty} h(y_n). \tag{2.10}$$ eq:01

Since h is strongly quasiconvex with modulus $\gamma > 0$,

$$\begin{aligned} & h(y_n) \leq \max\{h(x_1), h(x_n)\} - \frac{\gamma\|x_n\|}{2(1+\|x_n\|)^2}\|x_1 - x_n\|^2 \\ \Longrightarrow\ & \frac{h(y_n)}{\|x_n\|} \leq \max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{h(x_n)}{\|x_n\|}\right\} - \frac{\gamma}{2(1+\|x_n\|)^2}\|x_1 - x_n\|^2. \end{aligned} \tag{2.11}$$ for:compact

On the other hand, since $h(x_n) < M$ for every n, we have

$$\frac{h(y_n)}{\|x_n\|} \leq \max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{h(x_n)}{\|x_n\|}\right\} \leq \max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{M}{\|x_n\|}\right\}. \tag{2.12}$$ eq:02

Moreover, it follows from (2.10) and (2.12) that

$$\lim_{n\to+\infty} \frac{h(y_n)}{\|x_n\|} = 0.$$

Then, by taking $\liminf_{n\to+\infty}$ in (2.11) and using the last limit, we have

$$\begin{aligned} \liminf_{n\to+\infty}\left(\max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{h(x_n)}{\|x_n\|}\right\}\right) &\geq \frac{\gamma}{2}\left(\lim_{n\to+\infty} \frac{\|x_1 - x_n\|^2}{(1+\|x_n\|)^2}\right) \\ &= \frac{\gamma}{2}\underbrace{\left(\lim_{n\to+\infty} \frac{\left\|\frac{x_1}{\|x_n\|} - \frac{x_n}{\|x_n\|}\right\|^2}{\left(\frac{1}{\|x_n\|}+1\right)^2}\right)}_{=1}. \end{aligned} \tag{2.13}$$ for:P3

Therefore, it follows from (2.12) and (2.13) that

$$0 < \frac{\gamma}{2} < \liminf_{n\to+\infty}\left(\max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{h(x_n)}{\|x_n\|}\right\}\right) \leq \liminf_{n\to+\infty}\left(\max\left\{\frac{h(x_1)}{\|x_n\|}, \frac{M}{\|x_n\|}\right\}\right) = 0,$$

i.e., $\gamma = 0$, a contradiction. Therefore, h is coercive. □

Remark 2.1. *(i) Assumption (P0), which holds trivially in finite dimensional spaces and also when* $\operatorname{dom} h$ *is bounded, is a compactness-type assumption, extremely useful for providing existence results for noncoercive variational problems. Example of variational problems in which assumption (P0) holds may be found in [1, Chapter 15] among others.*

(ii) If in addition to the hypothesis of Proposition 2.1, K is closed and h is lsc, then the above proposition implies that $\operatorname{argmin}_K h$ *is a singleton by [4, Propositions 11.12, 11.8 and Theorem 11.10], improving [53, Theorem 3] for the case when h is strongly quasiconvex and the space is Hilbertian.*

Take a proper function $h : \mathcal{H} \to \overline{\mathbb{R}}$ and let $K \subseteq \mathcal{H}$ be a closed and convex set such that $K \subseteq \operatorname{dom} h$. The *proximal operator relative to* h *and* K *with parameter* $\beta > 0$ *at* $x \in \mathcal{H}$ is defined as:

$$\operatorname{Prox}_{\beta,h} : \mathcal{H} \rightrightarrows \mathcal{H},\ \operatorname{Prox}_{\beta,h}(K,x) = \operatorname*{argmin}_{y \in K} \left\{ h(y) + \frac{1}{2\beta}\|y - x\|^2 \right\}.$$

When $K = \mathcal{H}$, we write $\operatorname{Prox}_{\beta,h}(\cdot) := \operatorname{Prox}_{\beta,h}(\mathcal{H}, \cdot)$. If h is proper, lsc and convex, then $\operatorname{Prox}_{\beta,h}$ becomes a single-valued operator (see [4, Proposition 12.15]).

In the nonconvex case, we have the following result.

lemma:HL **Lemma 2.1.** ([15, Proposition 3.1]) *Let K be a closed and convex set in $\mathcal{H}$, $h : \mathcal{H} \to \overline{\mathbb{R}}$ be a proper, lsc, strongly quasiconvex function with modulus $\gamma \geq 0$ and such that $K \subseteq \operatorname{dom} h$, $\beta > 0$ and $x \in K$. If $\overline{x} \in \operatorname{Prox}_{\beta,h}(K,x)$, then*

$$h(\overline{x}) - \max\{h(y), h(\overline{x})\} \leq \frac{\lambda}{\beta}\langle \overline{x} - x, y - \overline{x}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \lambda\gamma\right)\|y - \overline{x}\|^2, \quad \forall\, y \in K,\ \forall\, \lambda \in [0,1]. \tag{2.14}$$

strong:formula

We close this section by recalling two well known definitions of monotonicity of a bifunction. Given a nonempty set C in $\mathcal{H}$ and a bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, f is said to be:

(i) monotone on C, if for all $x, y \in C$, we have

$$f(x,y) + f(y,x) \leq 0. \tag{2.15}$$

def:monotone

(ii) pseudomonotone on C, if for all $x, y \in C$, we have

$$f(x,y) \geq 0 \implies f(y,x) \leq 0. \tag{2.16}$$

def:pseudomonotone

Every monotone bifunction is pseudomonotone, but the reverse statement does not hold in general (see, for instance, [16]).

For more on generalized convexity and generalized monotonicity we refer to [3, 7, 16, 27, 28, 33, 49, 50] and references therein.

3 The Algorithm

sec:3

Let K be a closed and convex set in $\mathcal{H}$ and $f : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) is defined by

$$\text{find } \overline{x} \in K : \quad f(\overline{x}, y) \geq 0,\ \forall\, y \in K. \tag{EP}$$

EP

Its solution set is denoted by $S(K,f)$.

A problem closely related to Problem (EP) is the Minty (or dual) equilibrium one that is defined as

$$\text{find } \overline{z} \in K : \quad f(y, \overline{z}) \leq 0,\ \forall\, y \in K. \tag{dEP}$$

DEP

Its solution set is denoted by $S_d(K, f)$. Clearly, $\overline{x} \in S_d(K, f)$ if and only if, for every $x \in K$, $\overline{x} \in \bigcap_{y \in K} S_{f(x,y)}(f(x, \cdot))$.

Clearly, if f is pseudomonotone on K, then $S(K, f) \subseteq S_d(K, f)$. Conversely, if f is upper semicontinuous (usc henceforth) with respect to the first argument and convex with respect to the second argument (see for example [42]), then $S_d(K, f) \subseteq S(K, f)$. Furthermore, as was noted in [57, Lemma 1], this inclusion still holds true when f is semistrictly quasiconvex with respect to the second argument.

Before continuing, we mention the following example.

EP:motiv **Example 3.1.** *Mixed Variational Inequalities (MVI): Let $T : K \to \mathcal{H}$ be an operator and $h : K \to \mathbb{R}$ be a function. Then the (MVI) problem is defined by:*

$$\text{find } \overline{x} \in K : \quad \langle T(\overline{x}), y - \overline{x} \rangle + h(y) - h(\overline{x}) \geq 0, \ \forall \ y \in K. \tag{MVI}$$ MVI

Problem (MVI) *reduces to* (EP) *by taking* $f_h^T(x, y) = \langle T(x), y - x \rangle + h(y) - h(x)$.

Note that when $h \equiv 0$, problem (MVI) *reduces to the variational inequality problem, and that when $T \equiv 0$, problem* (MVI) *reduces to the minimization problem of h over K. But when h and T are both not* 0*, problem* (MVI) *presents a "conflict" between the operator and the function (see also [14, 21, 56]). This conflict is restricted to the case of nonconvex functions, because if h is convex, then* (MVI) *has a solution on any convex compact set K and various versions of PPA for solving it are well-known [35], but if h is not convex, then problem* (MVI) *may have no solutions even on convex and compact sets (see [21, page 127]).*

As a consequence of this conflict between the operator and the function, an assumption regarding the relationship between both them has to be imposed in order to ensure the existence of solutions for problem (MVI)*. Usually in the literature (see [14, 21, 23] and references therein), the authors use the mixed variational inequality property introduced in [56, Definition 3.1]).*

We will come back to this problem for applications and numerical experiments in Section 4.

Now, we present the assumptions on the equilibrium problem that will be used in the analysis of our algorithm.

($A0$) $f(x, x) = 0$ for all $x \in K$.

($A1$) f is continuous (jointly in both arguments) on an open set containing $K \times K$.

($A2$) f is pseudomonotone on K.

($A3$) For every $x \in K$, the function $f(x, \cdot)$ is strongly quasiconvex on K with modulus $\gamma > 0$.

($A3^*$) For every $y \in K$, if $\{t_k\}_k \subset \mathbb{R}, \{v_k\}_k \subset \mathcal{H}, t_k \to +\infty,\ v_k \rightharpoonup v$ and $\{f(y, t_k v_k)\}_k$ is bounded from above, then

$$\|v_k - v\| \to 0.$$

(*A*4) f satisfies the following Lipschitz condition: There exists $\eta > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - \eta(\|x-y\|^2 + \|y-z\|^2)\,,\ \forall\, x,y,z \in K. \quad (3.1)$$ Lips:cond

(*A*5) The Lipschitz constant η and the modulus of strong quasiconvexity γ are such that $12\eta < \gamma$.

We discuss now these assumptions.

rem:assumptions **Remark 3.1.** *(i) Under assumptions (Ai) with $i = 0, 1, 2, 3, 3^*$, problem* (EP) *has solutions by [22, Theorem 4.2]. Furthermore, under assumptions (Ai) with $i = 0, 1, 2, 3$, $S_d(K, f) = S(K, f)$, thus if $S(K, f) \neq \emptyset$ is a singleton, then $S_d(K, f) \neq \emptyset$ is a singleton too.*

(ii) As noted in [24, Remark 3.1(i)] in the finite-dimensional setting, assumption $(A0)$ is a direct consequence of $(A2)$ and $(A4)$.

(iii) Usually, the Lipschitz-type condition [39] (see also [51, 55, 57]) on f is stated as follows: there exist $\eta_1, \eta_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - \eta_1\|x-y\|^2 - \eta_2\|y-z\|^2,\ \forall\, x,y,z \in K. \quad (3.2)$$ Lips:cond1

We use assumption $(A4)$ instead of (3.2) *for simplicity. Note also that when $f(x,y) = \langle T(x), y-x\rangle$, with $T : K \to \mathcal{H}$ an L-Lipschitz continuous operator (for $L > 0$), relation* (3.1) *is fulfilled for $\eta = \frac{L}{2}$ (see [24, Remark 3.1(ii)] for instance).*

cond5:expla **Remark 3.2.** *(i) Assumption $(A5)$ was introduced for the first time in [24], which is (as far as we know) the first PPA for dealing with families of nonconvex equilibrium problems. Assumption $(A5)$ provides a relationship between the modulus of strong quasiconvexity of the second argument of f and its Lipschitz-type parameter. This assumption is particularly motivated by problem* (MVI) *when the involved function h is nonconvex and, as we explained before, this assumption mediates the conflict between the operator T and the function h, which in general could lead to equilibrium problems without solutions, even with compact feasible sets. We mention that the projected gradient method for variational inequalities for strongly monotone and Lipschitz continuous operators studied in [11, page 1109], requires a similar relationship between the modulus of strong monotonicity and the Lipschitz constant. We emphasize that although this assumption is inspired by problem* (MVI)*, our convergence analysis holds for any equilibrium problem satisfying our assumptions, beyond the* (MVI) *case.*

(ii) Note that assumption $(A5)$ implies that $\frac{1}{\gamma - 8\eta} < \frac{1}{4\eta}$, so that there exists $\varepsilon > 0$ such that $\left[\frac{1}{\gamma-8\eta} + \varepsilon, \frac{1}{4\eta} - \varepsilon\right]$ is a proper interval. We will choose the sequence of regularization parameters $\{\beta_k\}_k$ of our algorithm so that it is contained in this interval. This choice, together with the uniqueness of the solution of (EP)*, resulting from the strong quasiconvexity assumption, will allow for a more direct convergence analysis of our algorithm, when compared with other proximal point methods.*

The two-step proximal type algorithm that we propose is based on the version presented in [5, page 54].

ESQ **Algorithm 1** Two-step Proximal Algorithm for Nonconvex EP's **2PPA**

Step 0. (Initialization). Let $x^0 \in K$, $\varepsilon > 0$ and $\{\beta_k\}_k \subseteq \left[\frac{1}{\gamma - 8\eta} + \varepsilon, \frac{1}{4\eta} - \varepsilon\right]$, $k := 0$.

Step 1. Compute

$$y^k \in \operatorname{argmin}_{x \in K} \left\{ f(x^k, x) + \frac{1}{2\beta_k} \|x - x^k\|^2 \right\}. \tag{3.3}$$ step:x

Step 2. If $y^k = x^k$, then STOP, $\{x^k\} = S(K, f)$. Otherwise, find

$$x^{k+1} \in \operatorname{argmin}_{x \in K} \left\{ f(y^k, x) + \frac{1}{2\beta_k} \|x - x^k\|^2 \right\}. \tag{3.4}$$ step:x1

Step 3. Let $k \leftarrow k + 1$ and go to Step 1.

Remark 3.3. *(i) The parameter $\varepsilon > 0$, whose existence is ensured by Assumption (A5), and the choice of the regularized parameter $\{\beta_k\}_k$ within a closed interval are validated by virtue of Remark 3.2(ii).*

(ii) Steps 1 and 2 at Algorithm 1 are well defined under assumptions (Ai) with $i = 1, 3, 3^*$ in virtue of Proposition 2.1.

(iii) Our two-step algorithm can be seen as a predictor-corrector method, where Step 1 acts as a prediction step and Step 2 works as a correction step, in which the objective function is updated while trying to remain close enough to the previous iterate. It is related to the extragradient method for solving monotone variational inequalities (see [11, 55]), a particular case of equilibrium problems, where the one-step version may be divergent when the operator fails to be strongly monotone, but the first step still generates an auxiliary point y^k such that, applying the iteration operator at y^k, generates a "good" next iterate, which induces a sequence converging to a solution of the problem.

As a first result, we validate the stopping criterion of Algorithm 1 whose proof is omitted because parts (a) and (b) are exactly [24, Propositions 3.2 and 3.3], respectively, while part (c) is straightforward

stop:criteria **Proposition 3.1.** *Let K be a closed and convex set in $\mathcal{H}$, $f : K \times K \to \mathbb{R}$ be a bifunction, $\{\beta_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$ and $\{y^k\}_k$ be the sequences generated by Algorithm 1. Then:*

(a) *If Algorithm 1 stops at Step 2 of iteration k, then x^k is the unique solution of* (EP).

(b) *If $x^k \neq y^k$, then $f(x^k, y^k) < 0$.*

(c) *If $x^k \neq x^{k+1}$, then $f(y^k, x^{k+1}) < f(y^k, x^k)$.*

The ensuing result serves as a cornerstone in establishing the convergence of Algorithm 1 towards the solution of problem (EP).

prop:01 **Proposition 3.2.** *Let K be a closed and convex set in $\mathcal{H}$, $f : K \times K \to \mathbb{R}$ be a bifunction, $\{\beta_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ and $\{y^k\}_k$ be the sequences generated by Algorithm 1 and $S_d(K, f) = \{\overline{x}\}$. Suppose that assumptions (Ai) with $i = 1, 2, 3, 3^*, 4$ hold. Then for every $k \geq 0$, one of the following statements hold:*

(a) *If $f(y^k, x^{k+1}) \geq f(y^k, \overline{x})$ and $f(x^k, y^k) \geq f(x^k, x^{k+1})$, then*

$$\left(\frac{1 + \gamma\beta_k}{2}\right)(\|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - y^k\|^2) + \|x^k - y^k\|^2 \leq \|x^k - \overline{x}\|^2. \tag{3.5}$$ C1

(b) *If $f(y^k, x^{k+1}) < f(y^k, \overline{x})$ and $f(x^k, y^k) < f(x^k, x^{k+1})$, then*

$$\left(\frac{1 + \gamma\beta_k}{2}\right)\|x^{k+1} - \overline{x}\|^2 + \left(\frac{1 + \gamma\beta_k}{2} - 4\eta\beta_k\right)\|x^{k+1} - y^k\|^2 + (1 - 4\eta\beta_k)\|x^k - y^k\|^2 \leq \|x^k - \overline{x}\|^2. \tag{3.6}$$ C2

(c) *If $f(y^k, x^{k+1}) \geq f(y^k, \overline{x})$ and $f(x^k, y^k) < f(x^k, x^{k+1})$, then*

$$\left(\frac{1 + \gamma\beta_k}{2}\right)\|x^{k+1} - \overline{x}\|^2 + \left(\frac{1 + \gamma\beta_k}{2} - 4\eta\beta_k\right)\|x^{k+1} - y^k\|^2 + (1 - 4\eta\beta_k)\|x^k - y^k\|^2 \leq \|x^k - \overline{x}\|^2. \tag{3.7}$$ C3

(d) *If $f(y^k, x^{k+1}) < f(y^k, \overline{x})$ and $f(x^k, y^k) \geq f(x^k, x^{k+1})$, then*

$$\left(\frac{1 + \gamma\beta_k}{2} - 4\eta\beta_k\right)\left(\|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - y^k\|^2\right) + \|x^k - y^k\|^2 \leq \|x^k - \overline{x}\|^2. \tag{3.8}$$ C4

Proof. It follows from Step 2 and Lemma 2.1 that for all $k \geq 0$ one has

$$f(y^k, x^{k+1}) - \max\{f(y^k, y), f(y^k, x^{k+1})\} \leq \frac{\lambda}{\beta_k}\langle x^{k+1} - x^k, y - x^{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right)\|y - x^{k+1}\|^2.$$

Taking $y = \overline{x}$ with $\{\overline{x}\} = S_d(K, f)$, we have two cases.

(*i*) If $f(y^k, x^{k+1}) \geq f(y^k, \overline{x})$, then

$$0 \leq \langle x^{k+1} - x^k, \overline{x} - x^{k+1} \rangle + \frac{\beta_k}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|\overline{x} - x^{k+1}\|^2. \tag{3.9}$$

eq:S1

(*ii*) If $f(y^k, x^{k+1}) < f(y^k, \overline{x})$, then

$$\begin{aligned} f(y^k, x^{k+1}) - f(y^k, \overline{x}) \leq {} & \frac{\lambda}{\beta_k} \langle x^{k+1} - x^k, \overline{x} - x^{k+1} \rangle \\ & + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|\overline{x} - x^{k+1}\|^2. \end{aligned} \tag{3.10}$$

eq:S2

Moreover, from Step 1 and Lemma 2.1 yields

$$\begin{aligned} f(x^k, y^k) - \max\{f(x^k, y), f(x^k, y^k)\} \leq {} & \frac{\lambda}{\beta_k} \langle y^k - x^k, y - y^k \rangle \\ & + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|y - y^k\|^2. \end{aligned}$$

Setting $y = x^{k+1}$, we obtain other two cases:

(*iii*) If $f(x^k, y^k) \geq f(x^k, x^{k+1})$, then

$$0 \leq \langle y^k - x^k, x^{k+1} - y^k \rangle + \frac{\beta_k}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^{k+1} - y^k\|^2. \tag{3.11}$$

eq:S3

(*iv*) If $f(x^k, y^k) < f(x^k, x^{k+1})$, then

$$\begin{aligned} f(x^k, y^k) - f(x^k, x^{k+1}) \leq {} & \frac{\lambda}{\beta_k} \langle y^k - x^k, x^{k+1} - y^k \rangle \\ & + \frac{\lambda}{2} \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \|x^{k+1} - y^k\|^2. \end{aligned} \tag{3.12}$$

eq:S4

We analyze case by case:
(*a*): Using identity (2.1) in relation (3.9), we have

$$\begin{aligned} & \left(1 - \beta_k \left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma \right) \right) \|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - x^k\|^2 \leq \|x^k - \overline{x}\|^2, \\ \overset{\lambda = \frac{1}{2}}{\Longrightarrow} \quad & \left(\frac{1 + \gamma\beta_k}{2} \right) \|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - x^k\|^2 \leq \|x^k - \overline{x}\|^2. \end{aligned}$$

Similarly, using identity (2.1) in relation (3.11) and taking $\lambda = \frac{1}{2}$, we obtain

$$\left(\frac{1 + \gamma\beta_k}{2} \right) \|x^{k+1} - y^k\|^2 + \|x^k - y^k\|^2 \leq \|x^{k+1} - x^k\|^2.$$

By adding both inequalities, we obtain relation (3.5).

(b): Assume that (3.10) and (3.12) hold. Since $\overline{x} \in S_d(K,f)$, $f(y^k,\overline{x}) \leq 0$. Hence, the inequality in (3.10) becomes

$$\frac{\lambda}{\beta_k}\langle x^{k+1}-x^k, \overline{x}-x^{k+1}\rangle + \frac{\beta_k}{2}\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\|\overline{x}-x^{k+1}\|^2 \geq f(y^k,x^{k+1}),$$

which in turn, applying ($A4$) with $x=x^k$, $y=y^k$ and $z=x^{k+1}$ gives

$$\begin{aligned}&\frac{\lambda}{\beta_k}\langle x^{k+1}-x^k, \overline{x}-x^{k+1}\rangle + \frac{\beta_k}{2}\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\|\overline{x}-x^{k+1}\|^2 \\ &\geq f(x^k,x^{k+1})-f(x^k,y^k)-\eta\left(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\right). \qquad (3.13)\end{aligned}$$

eq:S5

Combining (3.13) and (3.12) we obtain

$$\begin{aligned}&\frac{\lambda}{\beta_k}\langle x^{k+1}-x^k, \overline{x}-x^{k+1}\rangle + \frac{\beta_k}{2}\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\|\overline{x}-x^{k+1}\|^2 + \frac{\lambda}{\beta_k}\langle y^k-x^k, x^{k+1}-y^k\rangle \\ &\geq -\frac{\lambda}{2}\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\|x^{k+1}-y^k\|^2-\eta\left(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\right),\end{aligned}$$

that is to say,

$$\begin{aligned}2\langle x^{k+1}-x^k, \overline{x}-x^{k+1}\rangle &\geq -\beta_k\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\left(\|\overline{x}-x^{k+1}\|^2+\|x^{k+1}-y^k\|^2\right) \\ &+2\langle y^k-x^k, y^k-x^{k+1}\rangle-\frac{2\eta\beta_k}{\lambda}\left(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\right). \qquad (3.14)\end{aligned}$$

eq:KR

Hence, it follows from (2.1) and (3.14) that

$$\begin{aligned}\|x^k-\overline{x}\|^2-\|x^{k+1}-\overline{x}\|^2 &\geq -\beta_k\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\left(\|\overline{x}-x^{k+1}\|^2+\|x^{k+1}-y^k\|^2\right) \\ &+\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2-\frac{2\eta\beta_k}{\lambda}\left(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\right),\end{aligned}$$

which implies

$$\begin{aligned}\|x^k-\overline{x}\|^2 &\geq \left(1-\beta_k\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\right)\|x^{k+1}-\overline{x}\|^2+\left(1-\frac{2\eta\beta_k}{\lambda}\right)\|x^k-y^k\|^2 \\ &+\left(1-\beta_k\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)-\frac{2\eta\beta_k}{\lambda}\right)\|x^{k+1}-y^k\|^2.\end{aligned}$$

By taking $\lambda=\frac{1}{2}$, we obtain (3.6).

(c): Using assumption ($A4$) with $x=x^k$, $y=y^k$ and $z=x^{k+1}$ in (3.12),

$$\begin{aligned}&\langle y^k-x^k, x^{k+1}-y^k\rangle+\frac{\beta_k}{2}\left(\frac{\lambda}{\beta_k}-\gamma+\lambda\gamma\right)\|x^{k+1}-y^k\|^2 \\ &\geq \frac{\beta_k}{\lambda}\left(-f(y^k,x^{k+1})-\eta\left(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\right)\right).\end{aligned}$$

Since $f(y^k, x^{k+1}) \leq 0$ by Proposition 3.1(b), we have

$$\langle y^k - x^k, x^{k+1} - y^k\rangle + \frac{\beta_k}{2}\left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right)\|x^{k+1} - y^k\|^2 \geq -\frac{\eta\beta_k}{\lambda}\left(\|x^k - y^k\|^2 + \|x^{k+1} - y^k\|^2\right). \quad (3.15) \quad \text{eq:S7}$$

Thus, using (2.1) in (3.15) and taking $\lambda = \frac{1}{2}$, we obtain

$$\left(\frac{1+\gamma\beta_k}{2} - 4\eta\beta_k\right)\|x^{k+1} - y^k\|^2 + (1 - 4\eta\beta_k)\|x^k - y^k\|^2 \leq \|x^{k+1} - x^k\|^2. \quad (3.16) \quad \text{ee1}$$

Likewise, applying (2.1) in (3.9), and taking $\lambda = \frac{1}{2}$, we obtain

$$\left(\frac{1+\gamma\beta_k}{2}\right)\|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - x^k\|^2 \leq \|x^k - \overline{x}\|^2. \quad (3.17) \quad \text{ee2}$$

Combining (3.16) and (3.17), we get (3.7).

(d): Using assumption ($A4$) with $x = y^k$, $y = x^{k+1}$ and $z = \overline{x}$ in (3.10),

$$\begin{aligned}
&\frac{\lambda}{\beta_k}\langle x^{k+1} - x^k, \overline{x} - x^{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta_k} - \gamma + \lambda\gamma\right)\|\overline{x} - x^{k+1}\|^2\\
&\geq f(y^k, x^{k+1}) - f(y^k, \overline{x})\\
&\geq -f(x^{k+1}, \overline{x}) - \eta\left(\|y^k - x^{k+1}\|^2 + \|\overline{x} - x^{k+1}\|^2\right). \qquad (3.18)
\end{aligned}$$

Since $\overline{x} \in S_d(K, f)$, $f(x^{k+1}, \overline{x}) \leq 0$. Hence, using (2.1) and taking $\lambda = \frac{1}{2}$,

$$\left(\frac{1+\gamma\beta_k}{2} - 4\eta\beta_k\right)\|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - x^k\|^2 - 4\eta\beta_k\|x^{k+1} - y^x\|^2 \leq \|x^k - \overline{x}\|^2. \quad (3.19) \quad \text{ee3}$$

Using (2.1) and taking $\lambda = \frac{1}{2}$ in (3.11), we have

$$\left(\frac{1+\gamma\beta_k}{2}\right)\|x^{k+1} - y^k\|^2 + \|x^k - y^k\|^2 \leq \|x^{k+1} - x^k\|^2. \quad (3.20) \quad \text{ee4}$$

Hence, (3.19) and (3.20) imply (3.8).

Therefore, for every $k \geq 0$, at least one among relations (3.5), (3.6), (3.7) and (3.8) holds. $\square$

In order to simplify the convergence analysis of Algorithm 1, we consolidate the four relations from Proposition 3.2 into a single relation in the following result.

prop:02 **Proposition 3.3.** *Let K be a closed and convex set in $\mathcal{H}$, $f : K \times K \to \mathbb{R}$ be a bifunction, $\{x^k\}_k$ and $\{y^k\}_k$ be the sequences generated by Algorithm 1 and*

and $S_d(K,f)=\{\overline{x}\}$. Suppose that assumptions (Ai) for $i=1,2,3,3^*,4$ holds. Then for every $k\geq 0$ we have

$$\left(\frac{1+\gamma\beta_k}{2}-4\eta\beta_k\right)\left(\|x^{k+1}-\overline{x}\|^2+\|x^{k+1}-y^k\|^2\right)+(1-4\eta\beta_k)\,\|x^k-y^k\|^2 \leq \|x^k-\overline{x}\|^2. \tag{3.21}$$

eq:key

Proof. Since $\frac{1+\gamma\beta_k}{2}\geq\frac{1+\gamma\beta_k}{2}-4\eta\beta_k$ and $1\geq 1-4\eta\beta_k$ for every $k\geq 0$, the result follows from Proposition 3.2. □

In the following result, we prove that the sequence $\{x^k\}_k$, generated by Algorithm 1, converges strongly to the unique solution of problem (EP). With this purpose, we recall that by Remark 3.1(i), under assumptions (Ai) with $i=0,1,2,3$, $S_d(K,f)=S(K,f)=\{\overline{x}\}$.

thm:main01

Theorem 3.1. *Let K be a closed and convex set in $\mathcal{H}$, $f:K\times K\to\mathbb{R}$ be a bifunction, $\{x^k\}_k$ and $\{y^k\}_k$ be the sequences generated by Algorithm 1 and $S_d(K,f)=\{\overline{x}\}$. Suppose that assumptions (Ai) with $i=0,1,2,3,3^*,4,5$ hold. Then*

$$\sum_{k=0}^{+\infty}\left((\gamma-8\eta)\beta_k-1\right)\left(\|x^{k+1}-\overline{x}\|^2+\|x^{k+1}-y^k\|^2\right)<+\infty, \tag{3.22}$$

eq:a1

$$\sum_{k=0}^{+\infty}(1-4\eta\beta_k)\,\|x^k-y^k\|^2<+\infty. \tag{3.23}$$

eq:a2

Furthermore, the following limits exist

$$\lim_{k\to+\infty}\|x^{k+1}-\overline{x}\|=\lim_{k\to+\infty}\|x^{k+1}-y^k\|=\lim_{k\to+\infty}\|y^k-x^k\|=0. \tag{3.24}$$

eq:6

As a consequence, the sequences $\{x^k\}_k$ and $\{y^k\}_k$ converges strongly to the unique solution $\overline{x}$.

Proof. Note that $\frac{1+\gamma\beta_k}{2}-4\eta\beta_k=1+\frac{(\gamma-8\eta)\beta_k-1}{2}$ for every $k\geq 0$. Since assumption $(A5)$ holds and $\beta_k\in[\frac{1}{\gamma-8\eta}+\epsilon,\frac{1}{4\eta}-\epsilon]$ for all $k\geq 0$, the inequality in (3.21) can be expressed as

$$\frac{(\gamma-8\eta)\beta_k-1}{2}\left(\|x^{k+1}-\overline{x}\|^2+\|x^{k+1}-y^k\|^2\right)+(1-4\eta\beta_k)\,\|x^k-y^k\|^2 \leq \|x^k-\overline{x}\|^2-\|x^{k+1}-\overline{x}\|^2.$$

Summing up from $k=0$ to $k=N$, we have

$$\sum_{k=0}^{N}\frac{(\gamma-8\eta)\beta_k-1}{2}\left(\|x^{k+1}-\overline{x}\|^2+\|x^{k+1}-y^k\|^2\right)+\sum_{k=0}^{N}(1-4\eta\beta_k)\,\|x^k-y^k\|^2 \leq \|x^0-\overline{x}\|^2-\|x^{N+1}-\overline{x}\|^2\leq\|x^0-\overline{x}\|^2. \tag{3.25}$$

eq:7

Letting $N \to +\infty$, we conclude that the two terms on the left side of (3.25) are finite, which proves relations (3.22) and (3.23).

Furthermore, since $(\gamma - 8\eta)\beta_k - 1 \geq (\gamma - 8\eta)\varepsilon > 0$ and $1 - 4\eta\beta_k \geq 4\eta\varepsilon > 0$, for every $k \geq 0$, it follows from item (a) that

$$\lim_{k \to +\infty} \left(\|x^{k+1} - \overline{x}\|^2 + \|x^{k+1} - y^k\|^2\right) = 0, \quad \text{and} \quad \lim_{k \to +\infty} \|x^k - y^k\|^2 = 0,$$

which imply the desired limits of (3.24) and, as a consequence, the sequences $\{x^k\}_k$ and $\{y^k\}_k$ are both bounded. Moreover, since $\lim_{k \to +\infty} \|x^{k+1} - \overline{x}\| = 0$, we conclude from (3.24), that both $\{x^k\}_k$ and $\{y^k\}_k$ converges strongly to $\overline{x}$, which completes the proof. □

We can say even more, namely that the sequence $\{x^k\}_k$ converges linearly to the solution of problem (EP), as we show next.

cor:main **Corollary 3.1.** *Let K be a closed and convex set in $\mathcal{H}$ and $f : K \times K \to \mathbb{R}$ be a bifunction. Suppose that assumptions (Ai) with $i = 0, 1, 2, 3, 3^*, 4, 5$ hold. Then the sequence $\{x^k\}_k$, generated by Algorithm 1, converges linearly to the unique solution of problem* (EP)*, with the asymptotic constant equal to* $[1 + (\gamma/2 - 4\eta)\epsilon]^{-1}$.

Proof. From (3.21), using the fact that $\beta_k \subseteq [\frac{1}{\gamma - 8\eta} + \epsilon, \frac{1}{4\eta} - \epsilon]$, we have for every $k \geq 0$ that

$$\left(\frac{2 + (\gamma - 8\eta)\epsilon}{2}\right)\|x^{k+1} - \overline{x}\|^2 \leq \left(\frac{1 + \gamma\beta_k}{2} - 4\eta\beta_k\right)\|x^{k+1} - \overline{x}\|^2 \leq \|x^k - \overline{x}\|^2,$$

$$\Longrightarrow \|x^{k+1} - \overline{x}\|^2 \leq \frac{2}{2 + (\gamma - 8\eta)\epsilon}\|x^k - \overline{x}\|^2.$$

Hence, based on this inequality and Theorem 3.1, we conclude that the sequence $\{x^k\}_k$ converges linearly to the unique point $\{\overline{x}\} = S(K, f)$ with the stated asymptotic constant. □

If we replace (3.4) with $x^{k+1} = y^k$, then Algorithm 1 extends [24, Algorithm 1] to Hilbert spaces. As a corollary of Theorem 3.1, we obtain [24, Theorem 3.1] in infinite-dimensional spaces, preserving strong convergence.

conver:solution **Corollary 3.2.** *Let K be a closed and convex set in $\mathcal{H}$, $f : K \times K \to \mathbb{R}$ be a bifunction such that assumptions (Ai) with $i = 0, 1, 2, 3, 3^*, 4, 5$ hold. Let $x^0 \in K$ be given and $\{x^k\}_k$ be the sequence generated by the following iterative rule, for $k \geq 0$,*

$$x^{k+1} \in \operatorname{argmin}\left\{f(x^k, x) + \frac{1}{2\beta_k}\|x^k - x\|^2 : \ x \in K\right\}.$$

Then $\{x^k\}_k$ converges linearly to the unique solution of problem (EP).

Remark 3.4. *A consequence of Corollary 3.2 is [15, Theorem 5.2] in Hilbert spaces, which establishes the linear convergence of the proximal point algorithm for minimizing a strongly quasiconvex function $h : \mathcal{H} \to \mathbb{R}$ on a closed, lsc and convex set K with f defined as $f(x, y) := h(y) - h(x)$, with $x, y \in K$.*

4 Applications and Numerical Experiments

sec:4

In this section, we present applications of our previous results in fractional programming.

4.1 Fractional Programming

subsec:4-1

Given a subset $K \subseteq \mathcal{H}$, and functions $h : \mathcal{H} \to \mathbb{R}$ and $g : \mathcal{H} \to \mathbb{R}$, we define the fractional minimization problem by

$$\min_{x\in K} \varphi(x) \;=\; \min_{x\in K} \frac{h(x)}{g(x)}, \tag{FMP}$$

FMP

with $g(x) \neq 0$ for all $x \in K$. This problem has been extensively studied in the literature (see [7, 19, 49, 50], among others) due to its concrete applications in various fields of mathematical sciences, particularly in economics. For instance, it plays a crucial role in productivity theory, where it addresses the maximization of return/risk or profit/cost and the minimization of cost/time, among other objectives.

It is important to note that, in general, problem (FMP) is not convex. For example, if h is convex and g is affine, then φ is semistrictly quasiconvex. Similarly, if h is non-negative and convex, and g is positive and concave, then φ is also semistrictly quasiconvex, as shown in [7, Theorem 2.3.8].

In cases where h is strongly convex and g is affine, φ exhibits a generalized convexity property stronger than semistrict quasiconvexity. The following result reveals a class of functions that are strongly quasiconvex.

prop:frac

Proposition 4.1. *Suppose $\varphi(x) = \frac{h(x)}{g(x)}$ as in* (FMP)*, where h is strongly convex with modulus $\gamma > 0$ over its domain which is a convex set, and g is finite, positive and bounded from above by M on* $\operatorname{dom} h$*. If any of the following conditions holds:*

- (a) *g is affine,*
- (b) *h is nonnegative and g is concave,*
- (c) *h is nonpositive and g is convex.*

Then φ is strongly quasiconvex with modulus $\gamma' := \frac{\gamma}{M} > 0$.

Proof. Since $g(x) > 0$ for all $x \in \operatorname{dom} h$, then $\operatorname{dom} \varphi = \operatorname{dom} h$.
(a): Assume g is affine and bounded by M, meaning $0 < g(x) \leq M$ for every $x \in \operatorname{dom}, \varphi$. Now, consider $x, y \in \operatorname{dom}, \varphi$ and assume, without loss of generality, that $\varphi(x) \geq \varphi(y)$. Then, for every $\lambda \in [0, 1]$, we have

$$\varphi(\lambda x + (1-\lambda)y) \leq \frac{\lambda h(x) + (1-\lambda)h(y) - \frac{\gamma}{2}\lambda(1-\lambda)\|x-y\|^2}{\lambda g(x) + (1-\lambda)g(y)}$$
$$= \varphi(x) + \frac{(1-\lambda)g(y)}{\lambda g(x) + (1-\lambda)g(y)}(\varphi(y) - \varphi(x)) - \frac{\gamma}{2}\frac{\lambda(1-\lambda)}{(\lambda g(x) + (1-\lambda)g(y))}\|x-y\|^2.$$

By settting $\gamma' = \frac{\gamma}{M} > 0$ and since $\varphi(x) \geq \varphi(y)$, we obtain

$$\begin{aligned}\varphi(\lambda x + (1-\lambda)y) &\leq \varphi(x) - \frac{\gamma}{2}\frac{\lambda(1-\lambda)}{(\lambda g(x) + (1-\lambda)g(y))}\|x-y\|^2 \\ &\leq \varphi(x) - \frac{\gamma'}{2}\lambda(1-\lambda)\|x-y\|^2, \qquad (4.1)\end{aligned}$$

eq::fracstrong

i.e., φ is strongly quasiconvex with modulus γ'.

(b): Assume that g is concave, positive and bounded by M. Take $x, y \in \operatorname{dom}\varphi$ and suppose without loss of generality that $\varphi(x) \geq \varphi(y)$. Since g is concave, we have for every $\lambda \in [0,1]$ that

$$\begin{aligned}& M \geq g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y) > 0 \\ \Longleftrightarrow\ & \frac{1}{M} < \frac{1}{g(\lambda x + (1-\lambda)y)} \leq \frac{1}{\lambda g(x) + (1-\lambda)g(y)}.\end{aligned}$$

Since g is nonnegative, it follows that

$$\begin{aligned}\varphi(\lambda x + (1-\lambda)y) &\leq \frac{h(\lambda x + (1-\lambda)y)}{\lambda g(x) + (1-\lambda)g(y)} \\ &\leq \frac{\lambda h(x) + (1-\lambda)h(y) - \frac{\gamma}{2}\lambda(1-\lambda)\|x-y\|^2}{\lambda g(x) + (1-\lambda)g(y)} \\ &= \varphi(x) + \frac{(1-\lambda)g(y)}{\lambda g(x) + (1-\lambda)g(y)}(\varphi(y) - \varphi(x)) \\ &\quad - \frac{\gamma}{2}\lambda(1-\lambda)\frac{1}{(\lambda g(x) + (1-\lambda)g(y))}\|x-y\|^2,\end{aligned}$$

and the result follows as in part (a).

(c): Analogous to (b). □

The following corollary is a direct consequence of Proposition 4.1 in finite dimensions, so its proof is omitted.

quad:frac

Corollary 4.1. *Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices, $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, $h(x) = \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha$ and $g(x) = \frac{1}{2}\langle Bx, x\rangle + \langle b, x\rangle + \beta$. Take $K = \{x \in \mathbb{R}^n : \ m \leq g(x) \leq M\}$, with $0 < m < M$, and $\varphi(x) = \frac{h(x)}{g(x)}$ for $x \in K$. Suppose that A is a positive definite matrix. If any of the following conditions holds:*

(a) *$B = 0$ (the null matrix),*

(b) *h is nonnegative on K and B is negative semidefinite,*

(c) *h is nonpositive on K and B is positive semidefinite,*

then φ is strongly quasiconvex with modulus $\gamma' := \frac{\lambda_{\min}(A)}{M} > 0$.

Remark 4.1. *Proposition 4.1 shows a class of functions which are strongly quasiconvex without being convex. Indeed, let us consider* $K = [1, 2]$*,* $h(x) = \frac{1}{2}x^2 - 1$ *and* $g(x) = x + 1$*. By Proposition 4.1(c), the function* $\frac{h}{g}$ *is strongly quasiconvex on* K *with modulus* $\gamma = \frac{1}{3}$*, but it is not convex on* $[1, 2]$*.*

In order to provide more concrete applications and numerical experiments, we consider the following type of equilibrium problems.

Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $T, F : K \to \mathbb{R}^n$ be two operators and $G : K \to \mathbb{R}$ be a real-valued function. Then the IMVI problem is given by:

$$\text{Find } x^* \in K : \quad \langle T(x^*), F(y) - F(x^*)\rangle + G(y) - G(x^*) \geq 0, \ \forall\, y \in K. \quad \text{(IMVI)}$$

IMVI

Problem (IMVI) is a broad formulation, which includes several optimization problems such as inverse variational inequalities (see [17, 18, 58]), mixed variational inequalities (see [43, 35]) and minimization problems, among others. Moreover, we point out that the extended linear-quadratic programming studied in [47] can be formulated as an (IMVI), too (see [17]).

In the following example, we exhibit families of IMVI's which satisfy our assumptions (Ai) with $i = 1, \ldots, 5$.

ex:IMVI **Example 4.1.** *Take* $A, A_1 \in \mathbb{R}^{n\times n}$*,* $b, b_1, c \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$*. Consider* $K := \{x \in \mathbb{R}^n : \ m \leq \langle c, x\rangle + \alpha \leq M\}$ *with* $0 < m < M$*. Let* $T : K \to K$*,* $F : K \to \mathbb{R}^n$ *and* $G : K \to \mathbb{R}$ *be the operators and function defined by*

$$T(x) := x; \quad F(x) := \frac{Ax + b}{c^T x + \alpha}, \quad G(x) := \frac{x^T A_1 x + b_1^T x + \beta}{c^T x + \alpha}. \tag{4.2}$$

ex:operators

By taking $f : K \times K \to \mathbb{R}$ *given by:*

$$f(x, y) := \left\langle x, \frac{Ay + b}{c^T y + \alpha} - \frac{Ax + b}{c^T x + \alpha}\right\rangle + \frac{y^T A_1 y + b_1^T y + \beta}{c^T y + \alpha} - \frac{x^T A_1 x + b_1^T x + \beta}{c^T x + \alpha}, \tag{4.3}$$

equi:bifunction

problem (IMVI) *reduces to an equilibrium problem.*

Let us check that f *defined in* (4.3) *satisfies the assumptions* (Ai) *with* $i = 1, \ldots, 5$ *under mild hypotheses. Indeed,* $(A0)$ *and* $(A1)$ *are straightforward. Furthermore, if* F *is monotone, then assumption* $(A2)$ *holds.*

For $(A3)$*: Note that* f *can be rewritten as*

$$\begin{aligned} f(x, y) &= \langle x, F(y) - F(x)\rangle + G(y) - G(x) \\ &= \frac{x^T(Ay + b)}{c^T y + \alpha} + \frac{y^T A_1 y + b_1^T y + \beta}{c^T y + \alpha} - \frac{x^T(Ax + b)}{c^T x + \alpha} - \frac{x^T A_1 x + b_1^T x + \beta}{c^T x + \alpha} \\ &= \frac{y^T A_1 y + (x^T A + b_1^T)y + b^T x + \beta}{c^T y + \alpha} - \frac{x^T(Ax + b)}{c^T x + \alpha} - \frac{x^T A_1 x + b_1^T x + \beta}{c^T x + \alpha}. \end{aligned}$$

Hence, by Proposition 4.1, for any $x \in K$ *the function* $f(x, \cdot)$ *is strongly quasiconvex on* K *when the matrix* A_1 *is symmetric positive definite, and its modulus of strong quasiconvexity is* $\gamma = \frac{\lambda_{\min}(A_1)}{M} > 0$*.*

For $(A4)$*: Since* F *is differentiable on* K*, it is Lipschitz-continuous. Let us denote by* L *its Lipschitz constant. Then, for every* $x, y, z \in K$*,*

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle x - y, F(y) - F(z) \rangle \\ &\geq -\frac{\sqrt{L}}{2}\|x-y\|^2 - \frac{1}{2\sqrt{L}}\|F(y) - F(z)\|^2 \\ &\geq -\frac{\sqrt{L}}{2}(\|x-y\|^2 + \|y-z\|^2). \end{aligned}$$

Therefore, $(A4)$ *holds.*

Finally, for $(A5)$*, we simply take the matrix* A_1 *with its smallest eigenvalue large enough so as to satisfy* $(A5)$*. This can be done because* η *depends on* A, b, c, α*, but does not depend on* A_1*.*

Remark 4.2. *Example 4.1 presents classes of functions whose sum is strongly quasiconvex. We recall that the sum of quasiconvex functions is not necessarily quasiconvex, even when one of the functions is linear (*$h_1(x) = \sqrt{|x|}$ *and* $h_2(x) = x$ *are quasiconvex but their sum is not). Other classes of quasiconvex functions whose sum is quasiconvex can be found in [3, 26].*

4.2 Numerical Experiments

subsec:4-2

We conducted numerical experiments to assess the performance of our proposed algorithm (referred to as **2PPA**) on the IMVI problem outlined in Example 4.1, varying sizes and inputs. The algorithm were implemented and executed in Python on an ASUS Laptop running Windows 11, equipped with an AMD Ryzen 7 5800H CPU and 16GB RAM. For stopping criteria, iterations halted either when $|x^k - y^k| < 10^{-5}$ or after exceeding 50 iterations. Furthermore, we use the SciPy.optimize package, function minimize, because, as it is unfortunately often the case when dealing with nonconvex functions, a closed form of the proximity operator of the involved function is not available.

To illustrate the numerical experiments, we tackled problem (IMVI) with $K := x \in \mathbb{R}^n : m \leq \langle c, x \rangle + \alpha \leq M$, where $0 < m < M$, and operators $T : K \to K$, $F : K \to \mathbb{R}^n$, and $G : K \to \mathbb{R}$ as defined in (4.2), with $A, A_1 \in \mathbb{R}^{n \times n}$, $b, b_1, c \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$.

In order to show numerical experiments, we consider problem (IMVI) with $K := \{x \in \mathbb{R}^n : m \leq \langle c, x \rangle + \alpha \leq M\}$ with $0 < m < M$, and $T : K \to K$, $F : K \to \mathbb{R}^n$ and $G : K \to \mathbb{R}$ are the operators and function defined in (4.2) with $A, A_1 \in \mathbb{R}^{n \times n}$, $b, b_1, c \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

Two numerical illustrations are given below.

Example 4.2. *We take* $n = 10$*,* $K = [0, 5]^n$ *and*

$$A = \begin{bmatrix} I_5 & 0_5 \\ 0_5 & 0_5 \end{bmatrix}, \quad A_1 = k \cdot I_{10},$$

$$b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}^T,$$

$$b_1 = k \cdot c, \quad \alpha = 26, \quad \beta = 0.$$

Clearly, for $x \in K$, $1 \leq c^T x + \alpha \leq 51$. Hence, for any $x \in K$, the function $f(x,.)$ is strongly quasiconvex with modulus $\gamma = \frac{k}{51}$. The operator F is Lipschitz continuous with constant $L = 1$. Therefore, we can choose k big enough so that our problem satisfies assumption $(A5)$. The algorithm ***2PPA*** *with starting point $x_0 = 5 * 1_{10}$ (the vector $(1, 1, \dots, 1) \in \mathbb{R}^{10}$) stops at*

$$x^8 = \begin{bmatrix} 3.71890868e-09 \\ 3.71890868e-09 \\ 3.71890873e-09 \\ 3.71890873e-09 \\ 3.71890870e-09 \\ 5.27593597e-01 \\ 5.27594242e-01 \\ 5.27593609e-01 \\ \\ 5.27595028e-01 \\ 5.27594556e-01 \end{bmatrix},$$

after 8 iterations.

We also evaluate the performance of this algorithm with two error criteria:

$$err_1 := \|x^k - y^k\|,$$

$$err_2 := -\min_{x \in K} f(y^k, x),$$

and report its behavior in the first experiment in Figure 1. Since x^* is a solution of our problem if and only if $f(x^*, x) \geq 0$ for any $x \in K$, the second criterion err_2 can characterize the accuracy of y^k as an approximate solution.

Example 4.3. *In the second experiment, each entry of the vectors b, c, b_1 and scalars α, β is randomly generated in the interval $(0, 1)$. The matrices A, A_1 are also randomly generated such that each entries of these matrices are nonnegative and, furthermore, the matrix A_1 is symmetric positive definite.*

We evaluate Algorithm **2PPA** *by using the following two errors:*

$$err_1 := \|x^k - y^k\|,$$

$$err_3 := \frac{\min_{x \in K} f(y^k, x)}{\min_{x \in K} f(x_0, x)}.$$

Since the input data are randomly generated, the initial value of $\min_{x \in K} f(x_k, x)$ can be very large, hence in this experiment we used err_3 instead of err_2 as in the last example. The error err_3 is a criterion which can describe how good y^k as an approximated solution compared with the initial point x^0.

We test **2PPA** *for different values of n and for each value, the average e-rrors and CPU time(s) for a family of 100 problems are reported in Table 1.*

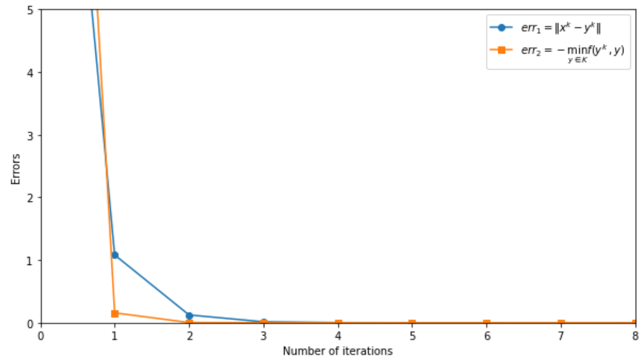


fig1

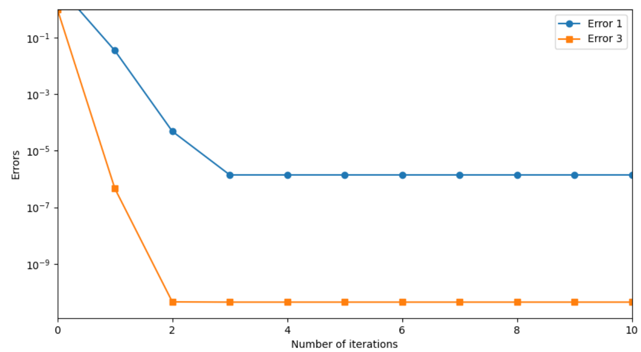
Figure 1: Behaviour of **2PPA** in the first experiment

n	err_1	err_3	CPU time (s)
2	3.010419122824038 e-06	2.2649807709768402 e-10	0.03693135738372802
5	2.308677924545267 e-06	22.8756516091227576 e-09	0.04780128002166748
10	2.215298250235035 e-06	3.6709888211066376 e-10	0.05515841007232667
20	1.3954276699983875 e-06	4.417886650676505 e-11	0.0462199878692627
50	5.873796558264606 e-07	8.366526945496797 e-12	0.04444539546966551
100	3.115897448989238 e-07	8.083427558321935 e-15	0.05987155914306641

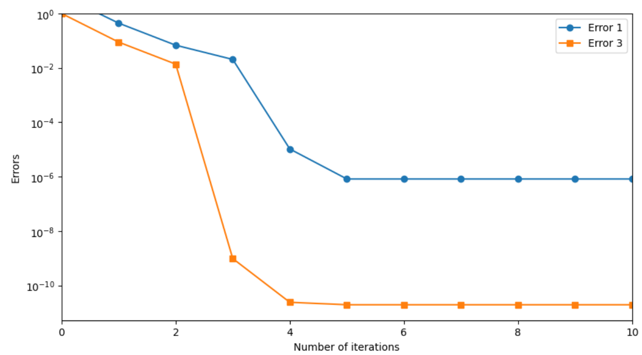
Table 1: Average errors and CPU time(s) of the **2PPA**

The behavior of **2PPA** *is illustrated in Figure 2 for dimension* $n = 20, 50, 100$. *Based on these 100 experiments, we observe that both errors go fast to zero and, both decrease very fast at the first iterations of the Algorithm and, furthermore, they are stable. As a consequence, we may conclude that Algorithm* **2PPA** *has a very good performance for this class of nonconvex equilibrium problems.*

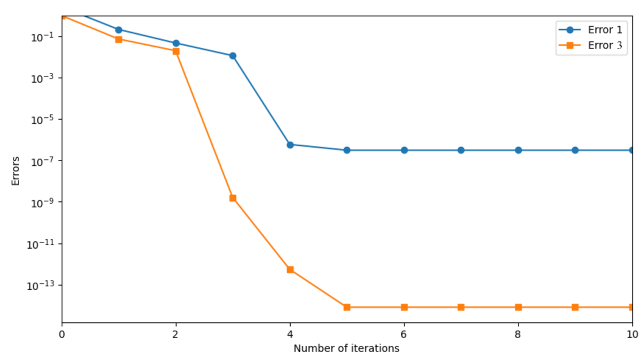
table 2



(a) $n = 20$



(b) $n = 50$



(c) $n = 100$

`fig 2` Figure 2: Average behavior of the **2PPA** for different values of n

Furthermore, in this experiment we also compare the performance of the **2PPA** *algorithm with other proximal point type algorithms available in the literature for nonconvex pseudomonotone equilibrium problems, namely* **PPA** *proposed in [24, Algorithm 1] and the extragradient algorithm proposed in [57,*

Algorithms	err_3	CPU time (s)
2PPA	1.0472139641978405 e-09	0.1423152327537537
PPA	2.5143330416663608 e-08	0.0826149344444275
EXTRA	0.006980008406946399	0.5858540773391725

(a) $n = 10$

Algorithms	err_3	CPU time (s)
2PPA	1.5035334837863297 e-11	0.1406812071800232
PPA	1.7711678199819674 e-10	0.1057134534062427
EXTRA	2.4484830741569943 e-06	0.6429274678230286

(b) $n = 20$

Algorithms	err_3	CPU time (s)
2PPA	5.496383152444126 e-12	0.100857675075531
PPA	1.0707509896410566 e-11	0.09209350347518923
EXTRA	0.30419149013148356	0.4192571043968201

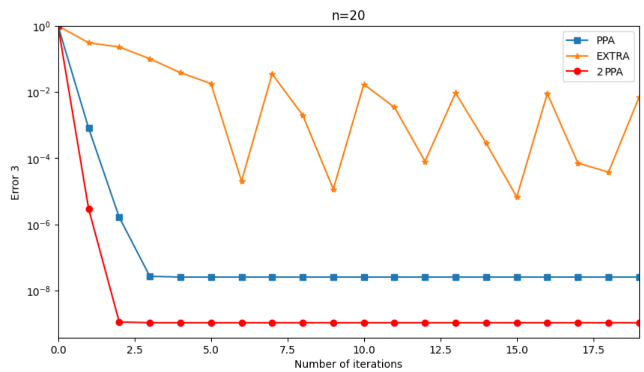
(c) $n = 50$

Table 2: Average errors and CPU time(s) of **2PPA**, **PPA**, **EXTRA** after 20 iterations

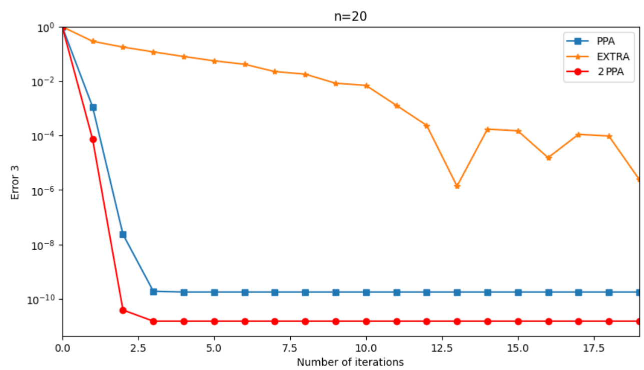
Algorithm 31] (denoted by **EXTRA***), for solving the* (IMVI) *problem.*

For each dimension $n = 10, 20, 50$, we randomly generate a family of 20 *problems. In Figure 3, the behavior of the error err_3 of these algorithms in the first* 20 *iterations is illustrated. Furthermore, we report in Table 2 the average CPU time(s) and the average error err_3 for these three algorithms after* 20 *iterations.*

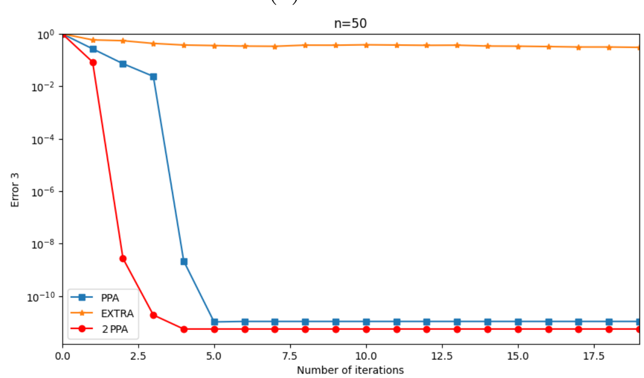
table 3



(a) $n = 10$



(b) $n = 20$



(c) $n = 50$

Figure 3: Average errors in the 20-first iterations of the **2PPA**, **PPA**, **EXTRA** for different values of n

fig 3

In view of these results, we observe that both **PPA** *and* **2PPA** *have a better behavior than the* **EXTRA** *algorithm in both error size and CPU time. As* *a consequence,* **2PPA** *seems to be a better acceleration procedure for proximal point type algorithms than* **Extra** *for dealing with this class of nonconvex equilibrium problems. Comparing now* **2PPA** *with* **PPA***, we observe that, with respect to both error size and CPU time one notices no significant difference, but, however, the error size of* **2PPA** *is better and the price to pay is given in the CPU time. Hence, both* **2PPA** *and* **PPA** *are competitive and* **2PPA** *improves in terms of the error value, at the cost of a slightly higher CPU time.*

Therefore, **2PPA** *seems to be a better acceleration method for nonconvex pseudomonotone equilibrium problems than* **EXTRA***. The error in* **2PPA** *decreases faster than* **PPA** *and becomes similar for a large iteration number. However, we observe that the error in* **2PPA** *is more stable than in* **PPA***. This is relevant in real problems because, in practice, the algorithm has to be stopped after an arbitrarily fixed number of iterations.*

5 Conclusions

We have proposed a two-step proximal point algorithm designed to solve pseudomonotone equilibrium problems in a Hilbert space, particularly, when the bifunction is strongly quasiconvex in its second argument. We prove strong convergence of the algorithm and a linear convergence rate under suitable assumptions. Furthermore, we provide new examples of strongly quasiconvex functions, as well as sufficient conditions for a quadratic fractional function to be strongly quasiconvex. Moreover, we illustrate our theoretical results with computational experiments on a certain class of equilibrium problems, beyond the realm of variational inequalities.

6 Declarations

6.1 Availability of supporting data

No data sets were generated during the current study. The used PYTHON codes are available from all authors on reasonable request.

6.2 Funding

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6.3 Authors' contributions

All authors contributed equally to the study conception, design and implementation, and wrote and corrected the manuscript.

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References

ABM [1] H. Attouch, G. Buttazzo, G. Michaille. “Variational Analysis in Sobolev and BV Spaces: Aplications to PDEs and Optimization”. MPS-SIAM, Philadelphia, (2006).

BBGT [2] C. Baiocchi, G. Buttazzo, F. Gastaldi, F. Tomarelli, General existence theorems for unilateral problems in continuum mechanics, *Arch, J. Rational Mech. Anal.*, **100**, 149–189 (1988).

BGJ [3] E.N. Barton, R. Goebel, R.R. Jensen, Functions which are quasiconvex under linear perturbations, *SIAM J. Optim.*, **22**, 1089–1105, (2012).

BC-2 [4] H.H. Bauschke, P.L. Combettes, “Convex Analysis and Monotone Operators Theory in Hilbert Spaces”. *CMS Books in Mathematics.* Springer-Verlag, second edition, (2017).

BCPP [5] G. Bigi, M. Castellani, M. Pappalardo, M. Passacantando. “Nonlinear Programming Techniques for Equilibria”. Springer, Switzerland, (2019).

BO [6] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63**, 123–145, (1994).

CM-Book [7] A. Cambini, L. Martein. “Generalized Convexity and Optimization: Theory and Applications”. Springer, (2009).

D-1959 [8] G. Debreu, “Theory of value”. John Wiley, New York, (1959).

CH [9] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6**, 117–136, (2005).

DSV [10] N.T.P. Dong, J.J. Strodiot, N.T.T. Van, V.H. Nguyen, A family of extragradient methods for solving equilibrium problems, *J. Ind. Manag. Optim.*, **11**, 619–630, (2015).

FP2 [11] F. Facchinei, J.-S. Pang. “Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. II”. Springer, New York, (2003).

FFB-SIAM [12] F. Flores-Bazán, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case *SIAM, J. Optim.*, **11**, 675–690, (2000).

FrS [13] J.B.G. Frenk, S. Schaible, Fractional programming. In: N. Hadjisavvas et al. (eds.): "Handbook of generalized convexity and generalized monotonicity". pp. 335–386, Springer-Verlag, Boston, (2005).

GL-2 [14] S.-M. Grad, F. Lara, Solving mixed variational inequalities beyond convexity, *J. Optim. Theory Appl.*, **190**, 565–580, (2021).

GLM-4 [15] S.-M. Grad, F. Lara, R.T. Marcavillaca, Properties and proximal point type method for strongly quasiconvex functions in Hilbert spaces, *Submitted*, (2023).

HKS [16] N. Hadjisavvas, S. Komlosi, S. Schaible, "Handbook of Generalized Convexity and Generalized Monotonicity". Springer-Verlag, Boston, (2005).

H [17] B. He, Algorithm for a class of generalized linear variational inequality and its application, *Sci China*, A, **25**, 939–945, (1995).

HHL [18] B. He, X.-Z. He, H.K. Liu, Solving a class of constrained "black-box" inverse variational inequalities, *Eur. J. Oper. Res.*, **204**, 391–401, (2010).

IL1 [19] A. Iusem, F. Lara, Second order asympotic functions and applications to quadratic programming, *J. Convex Anal.*, **25**, 1, 271–291, (2018).

IL3 [20] A. Iusem, F. Lara, Optimality conditions for vector equilibrium problems with applications. *J. Optim. Theory Appl.*, **180**, 187–206, (2019).

IL4 [21] A. Iusem, F. Lara, Existence results for noncoercive mixed variational inequalities in finite dimensional spaces, *J. Optim. Theory Appl.*, **183**, 122–138, (2019).

IL5 [22] A. Iusem, F. Lara, Quasiconvex optimization and asymptotic analysis in Banach spaces, *Optim.*, **69**, 2453–2470, (2020).

IL6 [23] A. Iusem, F. Lara, A note on "Existence results for noncoercive mixed variational inequalities in finite dimensional spaces", *J. Optim. Theory Appl.*, **187**, 607–608, (2020).

IL7 [24] A. Iusem, F. Lara, Proximal point algorithms for quasiconvex pseudomonotone equilibrium problems, *J. Optim. Theory Appl.*, **193**, 443–461, (2022).

IKS [25] A. Iusem, G. Kassay, W. Sosa, On certain conditions for the existence of solutions of equilibrium problems, *Math. Programm.*, **116**, 259–273, (2009).

JON [26] V. Jeyakumar, W. Oettli, M. Natividad, A solvability theorem for a class of quasiconvex mappings with applications to optimization. *J. Math. Anal. Appl.*, **179**, 537–546, (1993).

J-2 [27] M. Jovanović, A note on strongly convex and quasiconvex functions, *Math. Notes*, **60**, 584–585, (1996).

KL-2 [28] A. Kabgani, F. Lara, Strong subdifferentials: theory and applications in nonconvex optimization, *J. Global Optim.*, **84**, 349–368, (2022).

KHV [29] G. Kassay, T.N. Hai, N.T. Vinh, Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems, *J. Nonlinear Convex Anal.*, **19**, 959–986, (2018).

Ko [30] I.V. Konnov, Application of the proximal point method to nonmonotone equilibrium problems, *J. Optim. Theory Appl.*, **119**, 317–333, (2003).

Kor [31] G. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12**, 747-756, (1976).

KF [32] Ky Fan, A Minimax Inequality and Applications. In O. Shisha (ed.), "Inequality III", pp. 103–113. Academic Press, New York, (1972).

Lara-9 [33] F. Lara, On strongly quasiconvex functions: existence results and proximal point algorithms, *J. Optim. Theory Appl.*, **192**, 891–911, (2022).

Lara-8 [34] F. Lara, On nonconvex pseudomonotone equilibrium problems with applications, *Set-Valued Var. Anal.*, **30**, 355–372, (2022).

MAL [35] Y. Malitsky, Golden ratio algorithms for variational inequalities, *Math. Programm.*, **184**, 383–410, (2020).

M1 [36] B. Martinet, Regularisation d'inequations variationelles par approximations successives, *Rev. Francaise Inf. Rech. Oper.*, 154–159, (1970).

M2 [37] B. Martinet, Determination approchée d'un point fixe d'une application pseudo-contractante, *C.R. Acad. Sci. Paris*, **274**, 163–165, (1972).

MWG [38] A. Mas-Colell, M.D. Whinston, J.R. Green, "Microeconomic Theory", Oxford University Press, Oxford (1995).

Ma [39] G. Mastroeni, On auxiliary principle for equilibrium problems, *Publ. Dip. Math. dell, Universita di Pisa*, **3**, 1244–1258, (2000).

MT [40] A. Moudafi, M. Théra, Proximal and dynamical approaches to equilibrium problems, in Lecture Notes in Econom. Math. Systems **477**, M. Théra, T. Tichatschke (eds.), Springer, Berlin, 187–201, (1999).

Mo [41] A. Moudafi, Viscosity approximation methods for fixed-point problems, *J. Math. Anal. Appl.*, **241**, 46–55, (2000).

Muu [42] L.D. Muu, Stability property of a class of variational inequalities, *Optimization*, **15**, 347–351, (1984).

OG [43] N. Ovcharova, J. Gwinner, Semicoercive variational inequalities: From existence to numerical solutions of nonmonotone contact problems, *J. Optim. Theory Appl.*, **171**, 422–439, (2016).

P [44] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Soviet Math.*, **7**, 72–75, (1966).

QMH [45] T.D. Quoc, L.D. Muu, N.V. Hien, Extragradient algorithms extended to equilibrium problems, *Optimization*, **57**, 749–766, (2008).

rock-1976 [46] R.T. Rockafellar, Monotone operators and proximal point algorithms, *SIAM J. Control Optim.*, **14**, 877–898, (1976).

RW-SIAM1990 [47] R.T. Rockafellar, R. Wets, Generalized linear-quadratic problems of deterministic and stochastic optimal control in discrete time, *SIAM J. Control Optim.*, **28**, 810–820, (1990).

SV [48] M.V. Solodov, B.F. Svaiter, A new projection method for variational inequality problems, *SIAM J. Control Optim.*, **37**, 765–776, (1999).

Schaible [49] S. Schaible, Fractional programming, In: R. Horst and P. Pardalos (eds.), "Handbook of Global Optimization", pp. 495–608. Kluwer Academic Publisheds, Dordrecht, (1995).

Stancu [50] I.M. Stancu-Minasian. "Fractional Programming: Theory, Methods and Applications". Kluwer Academic Publishers, (1997).

VDN [51] D.Q. Tran, M.L. Dung, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems, *Optimization*, **57**, 749–776, (2008).

TaTa [52] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331**, 4506–515, (2007).

VNC-2 [53] A.A. Vladimirov, Ju.E. Nesterov, Ju.N. Chekanov, O ravnomerno kvazivypuklyh funkcionalah [On uniformly quasiconvex functionals], *Vestn. Mosk. un-ta, vycis. mat. i kibern.*, **4**, 18–27, (1978).

vNM [54] J. von Neumann, O. Morgenstern. "Theory of Games and Economic Behavior". Princeton University Press, (1944).

VSN [55] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems, *J. Optim. Theory Appl.*, **155**, 605–627, (2012).

WAN [56] M. Wang, The existence results and Tikhonov regularization method for generalized mixed variational inequalities in Banach spaces, *Ann. Math. Phys.*, **7**, 151–163, (2017).

YMU [57] L.H. Yen, L.D. Muu, An extragradient algorithm for quasiconvex equilibrium problems without monotonicity, *J. Global Optim.*, DOI: 10.1007/s10898-023-01291-y, (2023).

Z [58] Y.-B. Zhao, Iterative methods for monotone generalized variational inequalities, *Optimization*, **42**, 285–307, (1997).