A Subgradient Projection Method for Quasiconvex Multiobjetive Optimization

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ARTICLE HISTORY
Compiled November 27, 2024

ABSTRACT
We discuss a subgradient projection method for dealing with the nonconvex nonsmooth multiobjective optimization problem when every component of the vector-valued function is strongly quasiconvex in the sense of Polyak [29]. Under mild assumptions, we ensure that the generated sequence converges to an efficient solution point of the multiobjective optimization problem and we provide a kind of linear convergence rate. The algorithm is based on a specific generalized subdifferential that was recently introduced for strongly quasiconvex functions, this approach provides new and valuable information on the generated sequence and its convergent point. Finally, we present numerical experiments for classes of quadratic fractional programming problems.

KEYWORDS
Multiobjective optimization; Subgradient methods; Nonconvex optimization; Nonsmooth optimization; Generalized convexity; Quadratic fractional programming.

1. Introduction

Let K be a closed and convex set in $\mathbb{R}^n$ and $F := (f_1, \ldots, f_m) : K \to \mathbb{R}^m$ be a vector-valued function. Then the constrained multiobjective optimization problem is defined as

$$\min_{x \in K} F(x). \qquad \text{(MOP)}$$ MOP

It is said that a point $\overline{x} \in K$ is:

(i) a "weakly efficient" point of F (on K) if there is no other point $x_0 \in K$ such

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that

$$f_i(x_0) < f_i(\overline{x}),\ \forall\ i \in \{1, \ldots, m\}, \tag{1.1}$$

or equivalently, $(F(x_0) < F(\overline{x}))$;

(ii) a "efficient" point of F (on K) if there is no other point $x_0 \in K$ such that

$$\begin{aligned} &f_i(x_0) \leq f_i(\overline{x}),\ \forall\ i \in \{1, \ldots, m\}, \\ \&\ \ &\exists\ i_0 \in \{1, \ldots, m\} :\ f_{i_0}(x_0) < f_{i_0}(\overline{x}), \end{aligned} \tag{1.2}$$

or equivalently, $F(x) \leq F(\overline{x})$ and $F(x_0) \neq F(\overline{x})$;

(iii) a "ideal efficient" point of F (on K) if

$$f_i(\overline{x}) \leq f_i(x),\ \forall\ x \in K,\ \forall\ i \in \{1, \ldots, m\}. \tag{1.3}$$

Note that every ideal efficient point is efficient, and every efficient point is weakly efficient. The set of weakly efficient points is denoted by $E_W(F, K)$, the set of efficient points by $E(F, K)$, and the set of ideal efficient points by $I(F, K)$. Furthermore, if $m = 1$, then (MOP) coincides with the constrained minimization problem, and each kind of solution defined above coincides with the notion of a global minimum of the function on K.

Problem (MOP) is a significant and practical formulation in applied mathematics, known for its crucial applications in real-world scenarios across economics, finance, management, and engineering, among others. A notable example is the well-known portfolio selection problem [25]. Consequently, problem (MOP) has been extensively studied in the literature, focusing on existence results, optimality conditions, and iterative algorithms, particularly in the convex case. Concrete applications of multiobjective optimization problems appear in the design of electrical switching circuits, machine parts, airplanes, the configuration of industrial systems, as well as in non-cooperative games, see e.g. [16,23].

When dealing with a nonconvex problem (MOP), where the functions f_i are not convex, both the theoretical properties and the development of iterative algorithms become more challenging. Existence results and optimality conditions for the nonconvex case can be found in [7,8,14,20,24,30], while various iterative algorithms are discussed in [2,5,6,9,11,12,18,33].

In this paper, motivated by recent advances proposed in [21] and [4,19,22], we develop a subgradient projection method for problem (MOP) when each component f_i is strongly quasiconvex in the sense of [29] by using the strong subdifferential [19], which was specially introduced for strongly quasiconvex functions. Under mild assumptions, we ensure that the generated sequences converge to an efficient solution point of problem (MOP) and, moreover, we established a convergence rate which is almost linear. Furthermore, our developments reveal an intriguing and unexpected theoretical result, showing that the set of points $x \in K$ for which $F(x) \leq F(x^k)$ for all k (where $\{x^k\}_k$ is the sequence generated by the proposed algorithm) is a singleton when every component f_i of the vector-valued function F is strongly quasiconvex and lsc. These results extend and improve upon those from [4].

The structure of the paper is as follows: In Section 2, we introduce notation and basic definition from nonsmooth analysis, generalized convexity and multiobjective optimization theory. In Section 3, we present our proposed algorithm and the basic assumptions, we recall the existence and result and we discuss the convergence of the generated sequence. Under the usual assumptions for subgradient algorithms, we ensure that the generated sequence converges to an efficient solution of problem (MOP) and, furthermore, we established a kind of linear convergence result. Finally, in Section 4 we present applications and numerical experiments for quadratic fractional programming problems while in Section 5 we describe conclusions and future research lines for this subject.

2. Preliminaries and Basic Definitions

sec:2

The *inner product* of $\mathbb{R}^n$ and the *Euclidean norm* are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Given a convex and closed set $K \subseteq \mathbb{R}^n$, the *projection* of $x \in \mathbb{R}^n$ on K is denoted by $P_K(x)$ and satisfies that

$$v = P_K(u) \iff \langle u - v, w - v \rangle \leq 0, \ \forall \ w \in K. \tag{2.1}$$

proj_K

Given any $x, y, z \in \mathbb{R}^n$ and any $\beta \in \mathbb{R}$, the following relations holds:

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2. \tag{2.2}$$

iden:1

Given any extended-valued function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the effective domain of h is defined by $\operatorname{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. It is said that h is proper if $\operatorname{dom} h$ is nonempty and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$. The notion of properness is important when dealing with minimization problems.

It is indicated by $\operatorname{epi} h := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$ the epigraph of h, by $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ the sublevel set of h at the height $\lambda \in \mathbb{R}$ and by $\operatorname{argmin}_{\mathbb{R}^n} h$ the set of all minimal points of h. A function h is lower semicontinuous (lsc henceforth) at $\overline{x} \in \mathbb{R}^n$ if for any sequence $\{x_k\}_k \in \mathbb{R}^n$ with $x_k \to \overline{x}$, $h(\overline{x}) \leq \liminf_{k \to +\infty} h(x_k)$. Furthermore, the current convention $\sup_{\emptyset} h := -\infty$ and $\inf_{\emptyset} h := +\infty$ is adopted, as well as, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

A function h with convex domain is said to be

(a) convex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y), \ \forall \ \lambda \in [0,1], \tag{2.3}$$

def:convex

(b) strongly convex on $\operatorname{dom} h$ with modulus $\gamma \in]0, +\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \leq \lambda h(y) + (1-\lambda)h(x) - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2, \tag{2.4}$$

strong:convex

(c) quasiconvex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1-\lambda)y) \leq \max\{h(x), h(y)\}, \ \forall \ \lambda \in [0,1], \tag{2.5}$$

def:qcx

(d) strongly quasiconvex on $\operatorname{dom} h$ with modulus $\gamma \in]0, +\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \leq \max\{h(y), h(x)\} - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2. \tag{2.6}$$

strong:quasiconv

It is said that h is strictly convex (resp. strictly quasiconvex) if the inequality in (2.3) (resp. (2.5)) is strict whenever $x \neq y$ and $\lambda \in]0,1[$.

The relationship between all these notions is summarized below (we denote quasiconvex by qcx):

$$\begin{array}{ccccc} \text{strongly convex} & \Longrightarrow & \text{strictly convex} & \Longrightarrow & \text{convex} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{strongly qcx} & \Longrightarrow & \text{strictly qcx} & \Longrightarrow & \text{qcx} \end{array}$$

In general, all the reverse statements do not hold. For instance, the Euclidean norm $h_1(x) = \|x\|$ is strongly quasiconvex without being strongly convex on any bounded convex set (see [17, Theorem 2]), and the function $h_2(x) = \frac{x}{1+|x|}$ is strictly quasiconvex without being strongly quasiconvex on $\mathbb{R}$ while the other counterexamples are well-known.

Before continuing, let us show some new examples of strongly quasiconvex functions that are not convex.

rem:exam **Remark 1.** (i) Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function given by $h(x) = \sqrt{\|x\|}$. Clearly, h is nonconvex, but it is strongly quasiconvex on any $\mathbb{B}(0,r)$, $r > 0$, with modulus $\gamma = \frac{1}{5^{\frac{1}{4}} 2^{\frac{5}{4}} r^{\frac{1}{2}}}$ by [21, Theorem 17].

(ii) Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices, $a, b \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ be the function given by:

$$h(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha}{\frac{1}{2}\langle Bx, x\rangle + \langle b, x\rangle + \beta}. \tag{2.7}$$ qf

Take $0 < m < M$ and define:

$$K := \{x \in \mathbb{R}^n : \; m \leq g(x) \leq M\}.$$

If A is a positive definite matrix and at least one of the following conditions hold:

(a) $B = 0$ (the null matrix),

(b) f is nonnegative on K and B is negative semidefinite,

(c) f is nonpositive on K and B is positive semidefinite,

then h is strongly quasiconvex on K with modulus $\gamma = \frac{\lambda_{\min}(A)}{M}$ by [15, Corollary 4.1], where $\lambda_{\min}(A)$ is the minimum eigenvalue of A.

(iii) Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ be two strongly quasiconvex functions with modulus $\gamma_1, \gamma_2 > 0$, respectively. Then $h := \max\{h_1, h_2\}$ is strongly quasiconvex with modulus $\gamma := \min\{\gamma_1, \gamma_2\} > 0$ (straightforward).

(iv) Let $\alpha > 0$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a strongly quasiconvex functions with modulus $\gamma > 0$. Then αh is strongly quasiconvex with modulus $\gamma\alpha > 0$ (straightforward).

A proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be:

(i) 2-supercoercive, if

$$\liminf_{\|x\|\to+\infty} \frac{h(x)}{\|x\|^2} > 0, \tag{2.8}$$

(ii) coercive, if

$$\lim_{\|x\| \to +\infty} h(x) = +\infty. \tag{2.9}$$

or equivalently, if $S_\lambda(h)$ is bounded for all $\lambda \in \mathbb{R}$.

Clearly, every 2-supercoercive function is coercive, but the converse statement does not hold as the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \frac{|x|}{1+|x|}$ shows.

The following result is the starting point of our research.

ongqcx:coercive **Lemma 2.1.** ([21, Theorem 1]) *Let $K \subseteq \mathbb{R}^n$ be a convex set and $h : K \to \mathbb{R}$ be a strongly quasiconvex function with modulus $\gamma > 0$. Then h is 2-supercoercive.*

As a consequence, if K is closed and convex and h is lsc and strongly quasiconvex, then $\operatorname{argmin}_K h$ is a singleton (see [21, Corollary 3]).

Given a proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the convex subdifferential of h at $\overline{x} \in \operatorname{dom} h$ is defined by

$$\partial h(\overline{x}) := \{\xi \in \mathbb{R}^n : \ h(y) \geq h(\overline{x}) + \langle \xi, y - \overline{x} \rangle, \ \forall \ y \in \mathbb{R}^n\}, \tag{2.10}$$ subd:usual

and by $\partial h(x) = \emptyset$ if $x \notin \operatorname{dom} h$.

The convex subdifferential is an outstanding tool in continuous optimization. It is especially useful when the function is proper, lsc and convex, but when the function is nonconvex, the convex subdifferential is no longer useful. For this reason, several authors have introduced different generalized subdifferentials for dealing with different classes of nonconvex functions (see [26–28]).

For the case of quasiconvex and strongly quasiconvex functions, we recall the following notion of generalized subdifferential from [19].

The strong subdifferential of h at $\overline{x} \in \operatorname{dom} h \cap K$ is defined by

$$\begin{aligned} \partial^K_{\beta,\gamma} h(\overline{x}) := &\{\xi \in \mathbb{R}^n : \max\{h(y), h(\overline{x})\} \geq h(\overline{x}) + \frac{\lambda}{\beta}\langle \xi, y - \overline{x} \rangle \\ &+ \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta} - \lambda\gamma\right)\|y - \overline{x}\|^2, \ \forall \ y \in K, \ \forall \ \lambda \in [0,1]\}. \end{aligned} \tag{2.11}$$ strong:formula

Clearly, $\partial^K_{\beta,\gamma}$ is a closed and convex set. Furthermore, $\partial^K_{\beta,\gamma} h(\overline{x})$ is compact for every $\overline{x} \in \operatorname{int}(\operatorname{dom} h \cap K)$ by [19, Proposition 7].

Now, we recall some interesting and useful properties of the strong subdifferentials. Before that, we first recall that $\overline{x}$ is an (α, K)-strong minimum point of h if there exists $\alpha > 0$ such that

$$h(y) \geq h(\overline{x}) + \alpha\|y - \overline{x}\|^2, \ \forall \ y \in K \backslash \{\overline{x}\}, \tag{2.12}$$ eq:strongmin

Clearly, every (α, K)-strong minimum point is a strict minimum point and every strict minimum point is a global minimum point, but the reverse statements does not hold.

The strong subdifferential characterize strong minimum points for any proper function as we recall below.

char:min **Lemma 2.2.** ([19, Theorem 24(a)]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$, $\beta > 0$, $\gamma > 0$ and $\overline{x} \in K$. Then,*

$0 \in \partial^K_{\beta,\gamma} h(\overline{x})$ *iff* $\overline{x}$ *is a* $\left(\frac{\gamma^2\beta}{8(1+\gamma\beta)}, K\right)$*-strong minimum point of* h.

Other important properties for the strong subdifferential, when the function is strongly quasiconvex, are listed below.

subd:nonempty **Lemma 2.3.** ([19, Corollary 38(a)]) *Let* $K \subseteq \mathbb{R}^n$ *be a closed and convex set,* $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper and lsc function such that* $K \subseteq \operatorname{dom} h$*, and* $\beta > 0$*. If* h *is strongly quasiconvex on* K *with modulus* $\gamma > 0$*, then* $\partial^K_{\beta,\gamma} h(\overline{x}) \neq \emptyset$ *for every* $\overline{x} \in K$.

bounded:subd **Lemma 2.4.** ([4, Proposition 3.4]) *Let* $K \subseteq \mathbb{R}^n$ *be a closed and convex set and* $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper, continuous and strongly quasiconvex function on* $K \subseteq \operatorname{int}\operatorname{dom} h$ *with modulus* $\gamma > 0$*. If* K *is compact, then* $Y := \bigcup_{x\in K} \partial^K_{\beta,\gamma} h(x)$ *is nonempty and bounded.*

pro:KLx **Lemma 2.5.** ([19, Proposition 40]) *Let* $K \subseteq \mathbb{R}^n$ *be a closed and convex set,* $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper and lsc function such that* $K \subseteq \operatorname{dom} h$*,* $\beta > 0$ *and* $\overline{x} \in K$*. If* h *is strongly quasiconvex on* K *with modulus* $\gamma > 0$*, then*

$$y \in K \cap S_{h(\overline{x})}(h) \implies \langle \xi, y - \overline{x}\rangle \leq -\frac{\beta\gamma}{2}\|y - \overline{x}\|^2, \ \forall\, \xi \in \partial^K_{\beta,\gamma} h(\overline{x}). \tag{2.13}$$

To finalize this section, let us recall the following notion, which will be useful in the convergence analysis of our proposed algorithm.

def.quas **Definition 2.6.** Let $C \subseteq \mathbb{R}^n$ be nonempty set and $\{x^k\}_k \subseteq \mathbb{R}^n$ any sequence in $\mathbb{R}^n$. We say that,

(i) the sequence $\{x^k\}_k \subseteq \mathbb{R}^n$ is quasi-Fejér related to set $C \subseteq \mathbb{R}^n$ if for every $z \in C$ there exists a sequence $\{\epsilon_k\}_k \subseteq \mathbb{R}_+$ such that

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 + \epsilon_k,$$

with $\sum_{k=1}^{+\infty} \epsilon_k < +\infty$.

(ii) the sequence $\{x^k\}_k \subseteq \mathbb{R}^n$ converges to $\overline{x}$ quasi R-linearly, if there exists a constant $0 < \theta < 1$ and $\{\epsilon_k\}_k \subseteq \mathbb{R}_+$ (which does not depend on x_k) such that

$$\|x^{k+1} - z\|^2 \leq \theta\, \|x^k - z\|^2 + \epsilon_k,$$

with $\sum_{k=1}^{+\infty} \epsilon_k < +\infty$.

For a further study on nonsmooth analysis, generalized convexity and subgradient algorithms we refer to [1,2,9,10,17–19,21,26–29] and references therein.

3. A Subgradient Method

sec:3

Throughout the paper, we assume the following assumption on F:

$(A1)$ For every $i = 1, \ldots, m$, f_i is a proper, lsc and strongly quasiconvex function on $K \subseteq \operatorname{dom} f_i$ with modulus $\gamma_i > 0$.

Under assumption $(A1)$, the set $E(F, K)$ is nonempty as a consequence of [7, Theorem 2.1] and Lemma 2.1. The proof is provided for the sake of completeness.

sol:sets **Lemma 3.1.** *Let K be a closed and convex set in $\mathbb{R}^n$ and $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumption $(A1)$ holds. Then $E(F, K)$ is nonempty.*

Proof. Given $z \in K$, we consider $K(z) := \{x \in K : \ F(x) \leq F(z)\}$ and the scalarization problem

$$\min_{x \in K(z)} \sum_{i=1}^{m} f_i(x), \tag{3.1}$$ sc:problem

By [7, Theorem 2.1], $\overline{x} \in E(F, K)$ iff there exists $z(\overline{x}) \in K$ such that $\overline{x} \in \operatorname{argmin}_{K(z(\overline{x}))} \sum_{i=1}^{m} f_i$ (no convexity assumption is needed, see [7, Remark 2.1(i)]). Hence, we has to prove that problem (3.1) has solutions. Indeed, since assumption $(A1)$ holds, every f_i is lsc and strongly quasiconvex, i.e., lsc and 2-supercoercive by Lemma 2.1, thus the function $\sum_{i=1}^{m} f_i$ is lsc and 2-supercoercive too, i.e., $\operatorname{argmin}_{K(z)} \sum_{i=1}^{m} f_i$ is nonempty and compact. Thus, $E(F, K)$ is nonempty. □

Remark 2. The previous lemma can be easily extended to the case when every function f_i is lsc, strictly quasiconvex and coercive, with the same proof idea.

3.1. The Algorithm

In order to present our algorithm, we first consider the following compatibility conditions:

- $(C1)$ $K \subseteq \bigcap_{i=1}^{m} \operatorname{int} \operatorname{dom} f_i$.
- $(C2)$ There exists $M > 0$ such that $\partial^{K}_{\beta,\gamma_i} f_i(x) \subseteq \mathbb{B}(0, M)$ for every $x \in K$ and every $i = 1, \ldots, m$.
- $(C3)$ The sequences $\{\alpha_k\}_k \subseteq]0, +\infty[$ and $\{\rho_k\}_k \subseteq]0, 1[$ are such that $0 < \alpha_k \rho_k < \frac{M}{\gamma\beta}$ for every $k \in \mathbb{N}_0$, and

$$\sum_{k=0}^{\infty} \rho_k \alpha_k = +\infty, \quad \sum_{k=1}^{\infty} \rho_k \alpha_k^2 < +\infty, \tag{3.2}$$

  where $\gamma := \min_{1 \leq i \leq m}\{\gamma_i\} > 0$.

The algorithm that we considered is based in the version presented in [5] (see also [6,33]).

SQS **Algorithm 1** Subgradient Method for Strongly Quasiconvex Functions

Step 0. Take $\delta \in]0,1[$, $x^0 \in K$ and set $k := 0$.

Step 1. Given $x^k \in K$, for $i = 1, \ldots, m$ take $\xi_i^k \in \partial_{\beta,\gamma_i}^K f_i(x^k)$. Define

$$\xi_{max}^k \in \text{argmax}_{1\leq i\leq m}\|\xi_i^k\| \quad (\text{for E(F,K)}) \tag{3.3}$$

for:e

$$(\text{ or } \xi_{min}^k \in \text{argmin}_{1\leq i\leq m}\|\xi_i^k\| \quad \text{for } \text{E}_\text{W}(\text{F,K})). \tag{3.4}$$

for:ew

If $\xi_{max}^k = 0$, then STOP, x^k is an efficient solution of (MOP) (or also, if $\xi_{min}^k = 0$, then STOP, x^k is a weakly efficient solution of (MOP)). Otherwise, set $\xi_k := \xi_{max}^k$ and go to Step 2.

Step 2. Take $\rho_k \in]\delta, 1]$ and compute

$$x^{k+1} = (1-\rho_k)x^k + \rho_k P_K\left(x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k\right), \tag{3.5}$$

step:re-subd

where $\{\lambda_i^k\} \subseteq]0,1[$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \lambda_i^k = 1$.
Take $k = k+1$ and go to Step 1.

Before continuing, we analyze the previous assumptions below.

Remark 3. (*i*) Assumptions (Ci) with $i = 1,2,3$ are usual assumptions for subgradient algorithms. Indeed, $(C1)$ and $(C2)$ are actually used even for convex functions with the convex subdifferential (see Assumption 8.7(C) and Assumption 8.12 in [1], respectively). Moreover, as in the convex case, it follows from Lemmas 2.3 and 2.4 that assumption $(C2)$ is not too restrictive.

(*ii*) If $K = \mathbb{R}^n$, then Algorithm 1 can be viewed as [5, Algorithm (3.1)–(3.4)] for strongly quasiconvex functions. The convergence of the algorithm in [5] relies on the Lipschitz continuity assumption of the functions and the Gutirrez subdifferential. In contrast, our approach does not require Lipschitz continuity and instead employs the strong subdifferential, which is more suitable for strongly quasiconvex functions, as was pointed out in [4]. Regarding the projection step (3.5), depending on the structure of K– for instance, if K is a polyhedral set or a box –it can be computed using a closed-form formula.

(*iii*) If $\xi_{\min}^k = 0$, then there exists $i_0 \in \{1, \ldots, n\}$ such that $x^k \in \text{argmin}_K f_{i_0}$. Thus, $x^k \in E_W(F,K)$. On the other hand, if $\xi_{\max}^k = 0$, then we have $x^k \in I(F,K) \subseteq E(F,K)$. Therefore, when searching for an efficient solution using Algorithm 1 (or as referenced in [5,6,33]), the interesting scenario arises when Algorithm 1 never stops.

(*iv*) If $m = 1$, then Algorithm 1 reduces to the classical projected subgradient algorithm (with a relaxation step) for the scalar minimization problem. Given $x^k \in K$, find $\xi^k \in \partial_{\beta,\gamma}^K f(x^k)$ (where f is the scalar objective function) and define

$$x^{k+1} = (1-\rho_k)x^k + \rho_k P_K\left(x^k - \frac{\alpha_k}{\|\xi^k\|}\xi^k\right), \tag{3.6}$$

sgm

which encompasses [4, Algorithm 1] (including relaxation steps) for the minimization problem.

(v) Note that under assumption ($A1$), Algorithm 1 is well-defined. Indeed, for every $i \in \{1,\dots,m\}$ and every $k \in \mathbb{N}_0$, we have by Lemma 2.3 that the subgradient $\xi_i^k \in \partial^K_{\beta,\gamma_i} f_i(x^k)$ exists. Furthermore, under assumption ($C1$) and [19, Proposition 7(d)], we have for every iterate x^k and every $i \in \{1,\dots,m\}$ that the set $\partial^K_{\beta,\gamma_i} f_i(x^k)$ is bounded.

3.2. Convergence Analysis

For proving the convergence of the sequence generated for Algorithm 1, we consider the following set:

$$\Omega := \{x \in K : \, F(x) \le F(x^k), \ \forall \, k \in \mathbb{N}_0\}. \tag{3.7}$$

We emphasize that the analysis is given for efficient solutions, i.e., when ξ^k is chosen by relation (3.3). The analysis for weakly efficient solutions (by using relation (3.4)) is analogous, so it is omitted.

As a first result, we have.

first:result **Proposition 3.2.** *Let K be a closed and convex set in $\mathbb{R}^n$, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumption ($A1$) holds and $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Then, for every $k \in \mathbb{N}_0$, we have*

$$\|x^{k+1} - x^k\| \le \rho_k \alpha_k. \tag{3.8}$$ basic:rel

Proof. Indeed, it follows from (3.5) that

$$\begin{aligned}
\|x^{k+1} - x^k\| &= \rho_k \left\| P_K\left(x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k\right) - P_K(x^k)\right\| \\
&\le \rho_k \left\| x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k - x^k \right\| \\
&= \rho_k \frac{\alpha_k}{\|\xi_k\|}\left\| \sum_{i=1}^m \lambda_i^k \xi_i^k \right\| \\
&\le \rho_k \alpha_k,
\end{aligned}$$

and the result follows. □

To prove that the sequence generated by Algorithm 1 converges to an efficient solution, we assume the following condition.

($A2$) For all sequence $\{z^k\}_k \subseteq \mathbb{R}^n \backslash E(F, K)$, there exists $z \in \mathbb{R}^n$ such that $F(z) \le F(z^k)$ for $k = 0, 1, 2, \dots$

Regarding assumption ($A2$), we observe the following:

for:future **Remark 4.** (i) As was noted in [24], assumption ($A2$) is related to the completeness of the image of the vector-valued function F, that is, all nonincreasing sequences concerning $\mathbb{R}^m_+$ in the image of F have a lower bound. This is a standard assumption for ensuring the existence of Pareto (efficient) optimal points for vector optimization problems. Moreover, assumption ($A2$) is the usual assumption for optimization methods in convex and nonconvex multiobjective (and vector) optimization problems (see for instance [5,9,33] among others).

(ii) If $m = 1$, then assumption $(A2)$ follows immediately from $(A1)$, i.e., $(A2)$ is not needed. Indeed, since $m = 1$, problem (MOP) reduces to the scalar minimization problem, and since f_1 is proper, lsc and strongly quasiconvex on K, $\mathrm{argmin}_K h$ is nonempty by [21, Corollary 3].

Under assumption $(A2)$, the set $\Omega \neq \emptyset$ and, moreover, it is well-known that it has a unique optimal value, i.e., $F(x) = F(y)$ for all $x, y \in \Omega$ (see [5, Remark 4.2] for instance). However, under assumptions $(A1)$ and $(A2)$ we can say more as we show next.

unique:point

Proposition 3.3. *Let K be a closed and convex set in $\mathbb{R}^n$, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumptions $(A1)$ and $(A2)$ hold. If $\Omega \cap E(F, K) \neq \emptyset$, then $\Omega \cap E(F, K)$ is a singleton.*

Proof. By assumption $(A2)$, the set Ω is nonempty. Let us suppose by contradiction that there exists $x_1, x_2 \in \Omega \cap E(F, K)$ with $x_1 \neq x_2$, such that $F(x_1) = F(x_2)$, that is, $f_i(x_1) = f_i(x_2)$ for all $i = 1, \ldots, m$. Since K is convex, $\frac{1}{2}x_1 + \frac{1}{2}x_2 \in K$, and since every f_i is strongly quasiconvex with modulus $\gamma_i > 0$, we have

$$\begin{aligned} f_i\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) &\leq \max\{f_i(x_1), f_i(x_2)\} - \frac{\gamma_i}{8}\|x_1 - x_2\|^2 \\ &= f_i(x_1) - \frac{\gamma_i}{8}\|x_1 - x_2\|^2, \ \forall\, i \in \{1, \ldots, m\}. \end{aligned}$$

Since $x_1 \neq x_2$, $f_i\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) < f_i(x_1) = f_i(x_2)$ for all $i \in \{1, \ldots, m\}$, i.e., $x_1, x_2 \notin E(F, K)$, a contradiction. $\square$

The following relation for the iterates generated by Algorithm 1 holds.

key.ineq

Proposition 3.4. *Let K be a closed and convex set in $\mathbb{R}^n$, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumptions $(A1)$ and $(A2)$ hold, $\{\alpha_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1 and $\widetilde{x} \in \Omega$. Moreover, assume that the compatibility conditions $(C1)$, $(C2)$ and $(C3)$ also hold. Then, for every $k \in \mathbb{N}_0$, we have*

$$\|x^{k+1} - \widetilde{x}\|^2 + \frac{1 - \rho_k}{\rho_k}\|x^{k+1} - x^k\|^2 \leq \left(1 - \frac{\beta\alpha_k\rho_k}{\|\xi^k\|}\sum_{i=1}^{m}\gamma_i\lambda_i^k\right)\|x^k - \widetilde{x}\|^2 + \alpha_k^2\rho_k. \tag{3.9}$$

RT0

Proof. Set $z^k := P_K\left(x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^{m}\lambda_i^k\xi_i^k\right)$. Then we have from the update (3.5) that

$$x^{k+1} = (1 - \rho_k)x^k + \rho_k z^k \quad \text{and} \quad x^{k+1} - x^k = \rho_k(z^k - x^k). \tag{3.10}$$

RT

Now, using the fact that $\widetilde{x} \in \Omega$ ($\subseteq K$) and the nonexpansivity of the projection operator, we have

$$\begin{aligned}
\|z^k - \widetilde{x}\|^2 &= \left\| P_K\left(x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k\right) - P_K(\widetilde{x})\right\|^2 \\
&\le \left\| x^k - \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k - \widetilde{x}\right\|^2 \\
&= \|x^k - \widetilde{x}\|^2 - 2\left\langle x^k - \widetilde{x}, \frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k\right\rangle + \left\|\frac{\alpha_k}{\|\xi_k\|}\sum_{i=1}^m \lambda_i^k \xi_i^k\right\|^2 \\
&\le \|x^k - \widetilde{x}\|^2 + 2\frac{\alpha_k}{\|\xi^k\|}\sum_{i=1}^m \lambda_i^k \langle \widetilde{x} - x^k, \xi_i^k\rangle + \alpha_k^2. \qquad (3.11)
\end{aligned}$$

RT1

Since $\xi_i^k \in \partial^K_{\beta,\gamma} f_i(x^k)$ (for every $i = 1, \ldots, m$) and $\widetilde{x} \in \Omega$ ($f_i(\widetilde{x}) \le f_i(x)$ for all $x \in K$ and every $i = \ldots, m$), it follows from Lemma 2.5 that

$$\langle \widetilde{x} - x^k, \xi_i^k\rangle \le -\frac{\gamma_i \beta}{2}\|\widetilde{x} - x^k\|^2 \text{ for every } i = 1, \ldots, m.$$

From this and (3.11) we obtain

$$\|x^k - \widetilde{x}\|^2 - \|z^k - \widetilde{x}\|^2 \ge \frac{\beta\alpha_k}{\|\xi^k\|}\sum_{i=1}^m \gamma_i \lambda_i^k \|x^k - \widetilde{x}\|^2 - \alpha_k^2. \qquad (3.12)$$

RT2

On the other hand, using identity (2.2) and the first equality of (3.10)

$$||x^{k+1} - \widetilde{x}||^2 = (1-\rho_k)||x^k - \widetilde{x}||^2 + \rho_k||z^k - \widetilde{x}||^2 - \rho_k(1-\rho_k)||x^k - z^k||^2,$$

or equivalently,

$$\|x^{k+1} - \widetilde{x}\|^2 - \|x^k - \widetilde{x}\|^2 = \rho_k(\|z^k - \widetilde{x}\|^2 - \|x^k - \widetilde{x}\|^2) - \rho_k(1-\rho_k)\|x^k - z^k\|^2. \qquad (3.13)$$

RT3

Thus, combining (3.13) with (3.12), we deduce

$$\|x^{k+1} - \widetilde{x}\|^2 - \|x^k - \widetilde{x}\|^2 \le -\frac{\beta\alpha_k\rho_k}{\|\xi^k\|}\sum_{i=1}^m \gamma_i\lambda_i^k\|x^k - \widetilde{x}\|^2 + \alpha_k^2\rho_k - \rho_k(1-\rho_k)\|x^k - z^k\|^2.$$

which together with the second equality of (3.10) implies (3.9). □

rem:simple **Remark 5.** The inequality in (3.9) can be written as ($\gamma = \min_{1\le i\le m}\{\gamma_i\}$)

$$\|x^{k+1} - \widetilde{x}\|^2 + \frac{1-\rho_k}{\rho_k}\|x^{k+1} - x^k\|^2 \le \left(1 - \frac{\gamma\beta\alpha_k\rho_k}{\|\xi^k\|}\right)\|x^k - \widetilde{x}\|^2 + \alpha_k^2\rho_k. \qquad (3.14)$$

eq:0

In the following result, we show that the sequence $\{x^k\}_k$, generated by Algorithm 1, converges to a point in Ω whenever Ω is nonempty and, moreover, we show that Ω is a singleton.

convergence

Proposition 3.5. *Let K be a closed and convex set in $\mathbb{R}^n$, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumption $(A1)$ and $(A2)$ hold, $\{\alpha_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1 and $\gamma = \min_{1\le i\le m}\{\gamma_i\}$. Moreover, suppose that Assumption (Ci) with $i = 1, 2, 3$ holds. Then the following assertions hold:*

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1-\rho_k}{\rho_k}\|x^{k+1} - x^k\|^2 < +\infty$ *and* $\sum_{k=0}^{\infty} \alpha_k\rho_k\|x^k - \widetilde{x}\|^2 < +\infty$ *for any* $\widetilde{x} \in \Omega$.

(b) *For every $\widetilde{x} \in \Omega$, the limit $\lim_{k\to+\infty}\|x^k - \widetilde{x}\| = 0$, i.e., $x^k \to \widetilde{x} \in \Omega$. As a consequence, Ω is a singleton.*

Proof. (a): Let $\widetilde{x} \in \Omega$. Then by relation (3.14), we obtain

$$\frac{1-\rho_k}{\rho_k}\|x^{k+1} - x^k\|^2 + \frac{\gamma\beta\alpha_k\rho_k}{\|\xi^k\|}\|x^k - \widetilde{x}\|^2 \le \|x^k - \widetilde{x}\|^2 - \|x^{k+1} - \overline{x}\|^2 + \rho_k\alpha_k^2.$$

Summing up from $k = 0$ to $k = N$, we have

$$\begin{aligned}\sum_{k=0}^{N} \frac{1-\rho_k}{\rho_k}\|x^{k+1} - x^k\|^2 + \frac{\gamma\beta}{M}\sum_{k=0}^{N}\alpha_k\rho_k\|x^k - \widetilde{x}\|^2 & \\ &\le \|x^0 - \widetilde{x}\|^2 - \|x^{N+1} - \widetilde{x}\|^2 + \sum_{k=0}^{N}\rho_k\alpha_k^2 \\ &\le \|x^0 - \widetilde{x}\|^2 + \sum_{k=0}^{N}\rho_k\alpha_k^2.\end{aligned}$$

Letting $N \to +\infty$, and taking into account that $\sum_{k=0}^{+\infty}\rho_k\alpha_k^2 < +\infty$ by $(C2)$, we obtain (a).

(b): Now, since $\sum_{k=0}^{+\infty}\rho_k\alpha_k = +\infty$ by $(C2)$, we deduce from the second sum of (b) that $\lim_{k\to+\infty}\|x^k - \widetilde{x}\| = 0$, i.e., $x^k \to \widetilde{x} \in \Omega$. Hence, by uniqueness of the limit, Ω is a singleton.

□

Our main result, which shows that the sequence generated by Algorithm 1 converges to an efficient solution point of problem (MOP) whenever Ω is nonempty, is given below.

theo:main

Theorem 3.6. *Let K be a closed and convex set in $\mathbb{R}^n$, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumption $(A1)$ and $(A2)$ hold, $\{\alpha_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Moreover, suppose that Assumption (Ci) with $i = 1, 2, 3$ holds. Then the sequence $\{x^k\}_k$ converges to a efficient solution point of problem* (MOP).

Proof. By assumption $(A2)$ and Proposition 3.5(b), the sequence $\{x^k\}_k$ converges to the point $\widetilde{x} \in \Omega$. Suppose for the contrary that $\widetilde{x} \notin E(F, K)$. Then, there exists $x^* \in K$ such that $F(x^*) \le F(\widetilde{x})$ and $F(x^*) \ne F(\widetilde{x})$. Since $\widetilde{x} \in \Omega$, $f_i(\widetilde{x}) \le f_i(x^k)$ for all $i \in \{1, \dots, m\}$, thus

$$f_i(x^*) \le f_i(\widetilde{x}) \le f_i(x^k),\ \forall\, k \in \mathbb{N}_0,\ \forall\, i \in \{1, \dots, m\},$$

then $x^* \in \Omega$. By Proposition 3.5(b), Ω is a singleton, thus $x^* = \widetilde{x}$, a contradiction. Therefore, $\widetilde{x} \in E(F, K)$ and the proof is complete. □

Before continuing, we observe the following.

Remark 6. (i) The fact that Ω is a singleton is quite surprising. This may be partially explained by the especial structure of the solution sets of lsc strongly quasiconvex functions (which is a singleton) and also by both Remark 4 and Proposition 3.3.

(ii) As a consequence of Proposition 3.5(b), we have that if $m = 1$, then $\Omega = \operatorname{argmin}_K h$, which is a new information for the subgradient method proposed in [4].

We conclude this section with the following result, which establishes a form of linear convergence rate for Algorithm 1 in the sense of Definition 2.6. This result extends [4, Corollary 3.3] to the multiobjective setting.

Proposition 3.7. *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $F : K \to \mathbb{R}^m$ be a vector-valued function such that assumption ($A1$) and ($A2$) hold, $\{\alpha_k\}_k$ be a sequence of positive numbers, $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1 and $\gamma = \min_{1 \leq i \leq m}\{\gamma_i\}$. Suppose that assumptions (Ci) with $i = 1, 2, 3$ holds. Then the sequence $\{x^k\}_k$ converges quasi linearly to $\{\overline{x}\} = \Omega \cap E(F, K)$, and*

$$\|x^{k+1} - \overline{x}\|^2 \leq \prod_{j=0}^{k} \left(1 - \frac{\gamma\beta\delta\alpha_j}{M}\right) \|x^0 - \overline{x}\|^2 + \varepsilon_k, \tag{3.15}$$

eq:C

with $\lim_{k\to+\infty} \varepsilon_k = 0$.

Proof. The quasi-linear convergence is follows directly from (3.9) From (3.14), assumptions ($C2$) and ($C3$) and the fact that $\rho_k \in [\delta, 1]$. Moreover, for every $k \in \mathbb{N}_0$,

$$\|x^{k+1} - \overline{x}\|^2 \leq \left(1 - \frac{\gamma\beta\delta\alpha_k}{M}\right) \|x^k - \overline{x}\|^2 + \rho_k\alpha_k^2.$$

Since, for each $k \in \mathbb{N}_0$, we have $1 - \frac{\gamma\beta\delta\alpha_k}{M} \geq 1 - \frac{\gamma\beta\alpha_k\rho_k}{M} > 0$ by ($C3$), we obtain recursively

$$\begin{aligned}
\|x^{k+1} - \overline{x}\|^2 &\leq \left(1 - \frac{\gamma\beta\alpha_k\rho_k}{M}\right)\left(\left(1 - \frac{\gamma\beta\delta\alpha_{k-1}}{M}\right)\|x^{k-1} - \overline{x}\|^2 + \rho_{k-1}\alpha_{k-1}^2\right) + \rho_k\alpha_k^2 \\
&\ \vdots \\
&\leq \prod_{j=0}^{k}\left(1 - \frac{\gamma\beta\delta\alpha_k}{M}\right)\|x^0 - \overline{x}\|^2 + \sum_{j=0}^{k}\rho_j\alpha_j^2,
\end{aligned}$$

which implies (3.15) with $\varepsilon_k = \sum_{j=0}^{k}\rho_k\alpha_j^2$. □

4. Applications and Numerical Experiments

sec:04

In this section, we present numerical experiments to illustrate the performance of our proposed algorithm. Before that, we provide insights into how the strong subdifferential is calculated for strongly quasiconvex fractional functions

4.1. Quadratic Fractional Programming

subsec:4-1

Consider the function in (2.7) with $A \in \mathbb{R}^{n\times n}$ a symmetric matrix, $B = 0$ (the null matrix), $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then the quadratic fractional optimization problem consists of minimizing a ratio of two functions, and is defined by

$$\min_{x\in K} \frac{\frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha}{\langle b, x\rangle + \beta}. \tag{4.1}$$

frac:scalar

This problem has been extensively studied due to its economic applications, such as minimizing cost/time or maximizing return/ risk. The literature on this subject is vast (see [3,13,14,31,32] and references therein).

Now, let us consider

$$h_i(x) = \max_{j\in J} \left\{ \frac{\frac{1}{2}\langle A^i_j x, x\rangle + \langle a^i_j, x\rangle + \alpha^i_j}{\langle b^i_j, x\rangle + \beta^i_j} \right\}, \tag{4.2}$$

eq:const

with $A^i_j \in \mathbb{R}^{n\times n}$ symmetric and definite positive, $a^i_j, b^i_j \in \mathbb{R}^n$ and $\alpha^i_j, \beta^i_j \in \mathbb{R}$ for all $i \in \{1, \dots, m\}$ and all $j \in J$. Take fixed $M_2 > M_1 > 0$. We choose the feasible set K by

$$K := \{x \in \mathbb{R}^n : M_1 \leq \langle b^i_j, x\rangle + \beta^i_j \leq M_2, \ \forall\, i \in \{1, \dots, m\}, \ \forall\, j \in J\},$$

which is convex and compact.

By Remark 1(*ii*), every h_i is the maximum of $|J| < +\infty$ strongly quasiconvex functions, i.e., every h_i is strongly quasiconvex by Remark 1(*iii*) with modulus $\gamma_i = \frac{\min_{j\in J}\{\lambda_{\min}(A^i_j)\}}{M_2} > 0$.

Therefore, we aim to find a solution of the following quadratic fractional multiobjective optimization problem (QFMOP):

$$\min_{x\in K} H(x) := (h_1(x), \dots, h_m(x)). \tag{QFMOP}$$

QFMOP

It is important to note that, when $|J| \geq 2$, the functions h_i are not necessarily differentiable as they are the maximum of functions. Hence, we will need to compute the strong subdifferential for each h_i, since they are nonsmooth and strongly quasiconvex functions.

To address this, and considering the structure of the problem, we recall the following result, which provides a straightforward method to estimate the strong subdifferential in the quadratic fractional case. This result will be useful for numerical experiments.

prop:frac2

Proposition 4.1. ([22, Proposition 4.2]) *Suppose that $h(x) = \frac{f(x)}{g(x)}$ for all $x \in \operatorname{dom} h$, where f is strongly convex with modulus $\gamma > 0$, g is affine, positive, finite, and bounded* *from above by $\mu > 0$ on $\operatorname{dom} h$, and $\operatorname{dom} h$ is convex. Then for any $\rho > 0$ and $x_0 \in \operatorname{dom} h$, we have*

$$\frac{\rho}{\mu}\partial(f - \alpha g)(x_0) \subseteq \partial^{\operatorname{dom} h}_{\rho,\frac{\gamma}{\mu}} h(x_0),$$

with $\alpha := h(x_0)$.

A direct consequence of Proposition 4.1 is the following corollary; thus, its proof is omitted.

prop:frac3

Corollary 4.2. *Let A be a positive definite symmetric matrix, a, $b \in \mathbb{R}^n$ and α, $\beta \in \mathbb{R}$, $f(x) = \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha$, and $g(x) = \langle b, x\rangle + \beta$. Take $K := \{x \in \mathbb{R}^n : M_1 \leq g(x) \leq M_2\}$, with $0 < M_1 < M_2$, $\rho > 0$, and $h(x) = \frac{f(x)}{g(x)}$ for $x \in K$. Then for any $x_0 \in K$, we have*

$$\frac{\rho}{M_2}(Ax_0 + a - \alpha b) \in \partial^K_{\rho,\frac{\gamma}{M_2}} h(x_0),$$

with $\alpha := h(x_0)$ and $\gamma = \lambda_{\min}(A)$.

The following calculus rule will be useful for the numerical experiments.

maximum:func

Proposition 4.3. *Let $h(x) = \max_{j\in J} h_j(x)$, where $h_j : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function for all $j \in J$. Let $\beta > 0$, $\gamma \geq 0$, $K \in \mathbb{R}^n$ and $x \in K \cap (\cap_{j\in J} \operatorname{dom} h_j)$. If $\ell \in J$ is any index such that $h_\ell(x) = h(x)$ and $\xi \in \partial^K_{\beta,\gamma} h_\ell(x)$, then $\xi \in \partial^K_{\beta,\gamma} h(x)$.*

Proof. Since $\xi \in \partial^K_{\beta,\gamma} h(x)$, it follows that

$$\max\{h_\ell(y), h_\ell(x)\} \geq h_\ell(x) + \frac{\lambda}{\beta}\langle \xi, y - x\rangle + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta} - \lambda\gamma\right)\|y - x\|^2, \quad \forall\, y \in K,\ \forall\, \lambda \in [0, 1],$$

which, together with the definition of h and $h_\ell(x) = h(x)$ implies that

$$\begin{aligned}\max\{h(y), h(x)\} &= \max\{\max_{j\in J} h_j(y), \max_{j\in J} h_j(x)\} \geq \max\{h_\ell(y), h_\ell(x)\} \\ &\geq h(x) + \frac{\lambda}{\beta}\langle \xi, y - x\rangle + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta} - \lambda\gamma\right)\|y - x\|^2,\ \forall\, y \in K,\ \forall\, \lambda \in [0, 1].\end{aligned}$$

Therefore, $\xi \in \partial^K_{\beta,\gamma} h(x)$. □

In the particular case of the maximum of quadratic fractional functions, we have the following estimate for the strong subdifferential.

prop:frac4

Corollary 4.4. *Let $h(x) = \max_{j\in J}\left\{\frac{\frac{1}{2}\langle A_j x, x\rangle + \langle a_j, x\rangle + \alpha_j}{\langle b_j, x\rangle + \beta_j}\right\}$, with $A_j \in \mathbb{R}^{n\times n}$ symmetric and definite positive, $a_j, b_j \in \mathbb{R}^n$ and $\alpha_j, \beta_j \in \mathbb{R}$ for all $j \in J$. Take $\rho > 0$, $x_0 \in K$, with $K := \{x \in \mathbb{R}^n : M_1 \leq \langle b_j, x\rangle + \beta_j \leq M_2,\ \forall\, j \in J\}$, $0 < M_1 < M_2$. If $\ell \in J$ is any index such that $h_\ell(x) = h(x)$, then*

$$\frac{\rho}{M_2}(A_\ell x_0 + a - \alpha b) \in \partial^K_{\rho,\frac{\gamma}{M_2}} h(x_0),$$

with $\alpha := \frac{f_\ell(x_0)}{g_\ell(x_0)}$, $f_\ell(x) = \frac{1}{2}\langle A_\ell x, x\rangle + \langle a_\ell, x\rangle + \alpha_j$, $g_\ell(x) = \langle b_\ell, x\rangle + \beta_\ell$, *and* $\gamma = \min_{j\in J}\{\lambda_{\min}(A_j)\}$.

subsec:4-2

4.2. Numerical Experiments

In this section, we will provide some numerical illustrations as an application of Algorithm 1 for solving Problem (QFMOP) defined in the previous subsection. The algorithm was coded in Matlab R2016a and run on a PC Intel(R) Core(TM) i5-2430M CPU @ 2.40 GHz 4GB Ram. We used the Optimization Toolbox (fmincon, quadprog) to solve strongly convex and quadratic programming problems. We stopped the program by using the stopping criterion

$$\|x^{k+1} - x^k\| \le \epsilon.$$

Example 4.5. In the first experiment, we take $|J| = 2$, $m = 5$, $A_j^i = (i+j)I_{10}$, $a_j^i = (i+j-3)e$, $\alpha_j^i = -i-j+3$, $b_j^i = (i+j)e$, $M_1 = 1$, $M_2 = 5$, and $\beta_j^i = 4 + \frac{i}{i+j}$, where I_{10} is an identity matrix of order 10 and $e = (1,\dots,1)$. Clearly, $0 \in K$.

We test our algorithm with the initial point and parameters: $x_0 = (0,\dots,0)^T$, $\beta = 1$, $\lambda_i = 1/5$ for all $i = 1,\dots,5$. We consider the stopping criterion $\|x^{k+1} - x^k\| \le 10^{-5}$.

Figures 1 and 2 show the convergence results of $\log_{10}\|x^k - x^{k-1}\|$ with respect to the number of iterations and CPU time, respectively, where $n = 10$, $\rho_k = \frac{1}{2}$ and $\alpha_k = \frac{1}{100k+1}$ for all $k \ge 0$.

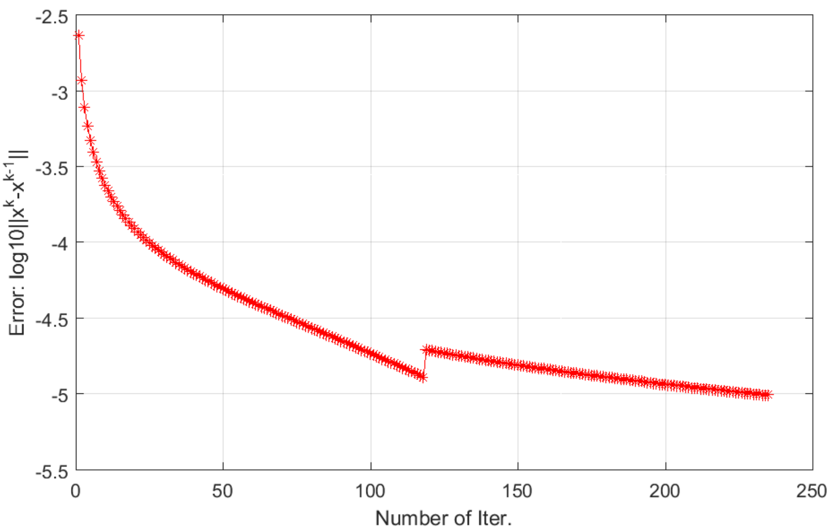

Figure 1. Convergence of $\log_{10}|x^k - x^{k-1}|$ of Algorithm 1 with respect to the number of iterations fig1

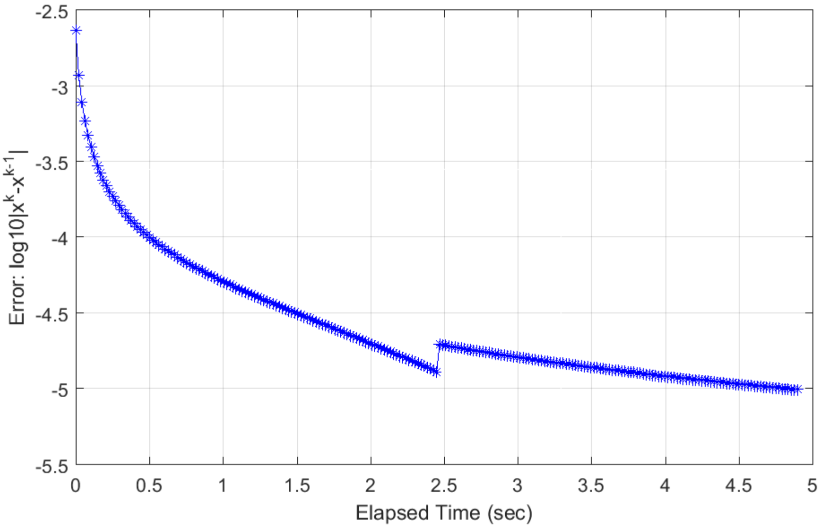

Figure 2. Convergence of $\log_{10}|x^k - x^{k-1}|$ of Algorithm 1 with respect to the CPU time fig2

The computation results with different parameters ρ_k and α_k are shown in **Table 1** with $n = 10$. The results depend on the dimension of $\mathbb{R}^n$ and reported in **Table 2**.

table1

	Parameters		Algorithm 1	
Init.point x^0	ρ_k	α_k	Iterations.	CPU time(s)
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{100k+1}$	235	5.0388
$(0,0,\dots,0)^\top$	$\frac{2}{3}$	$\frac{1}{100k+1}$	312	6.3024
$(0,0,\dots,0)^\top$	$\frac{3}{4}$	$\frac{1}{100k+1}$	351	7.7064
$(0,0,\dots,0)^\top$	$\frac{4}{5}$	$\frac{1}{100k+1}$	374	7.2696
$(0,0,\dots,0)^\top$	$\frac{5}{6}$	$\frac{1}{100k+1}$	390	7.4256
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{k+1}$	15	0.4524
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{5k+1}$	41	0.9828
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{20k+1}$	1081	20.7793
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{40k+1}$	517	9.6877
$(0,0,\dots,0)^\top$	$\frac{1}{2}$	$\frac{1}{80k+1}$	293	5.7408

Table 1. The comparative results for different parameters.

Based on this table, we see that the convergence rate of the algorithm depends on the control parameters.

n	Algorithm 1	
	Iter.	CPU. time(s)
10	235	4.5552
20	331	6.5052
30	454	8.7673
50	539	10.8421
70	318	7.7376
80	463	11.1073
100	620	12.5769

Table 2. Results for different dimensions n table2

Example 4.6. In this experiment, we take $|J| = 2$, $m = 5$, $M_1 = 1$, $M_2 = 5$, and $\beta^i_j = 4 + \frac{i}{i+j}$. Each entry of the vectors a^i_j, b^i_j and scalars α^i_j is randomly generated in the interval $[-n, n]$ for all $i = 1, \dots, 5$ and all $j = 1, 2$. The symmetric positive definite matrices A^i_j (with $i = 1, \dots, 5$ and $j = 1, 2$) are randomly generated by using commends $Q = H * D * H'$, $H = orth(randn(n))$, $D = diag(abs(randn(n, 1)) + 0.3)$. We test our algorithm with the initial point and parameters: $x_0 = (0, \dots, 0)^T$, $\beta = 1$, $\lambda_i = 1/5$ for all $i = 1, \dots, 5$, $\rho_k = \frac{1}{2}$, and $\alpha_k = \frac{1}{100k+1}$ for all $k \geq 0$. We consider the stopping criterion $\|x^{k+1} - x^k\| \leq 10^{-5}$.

Table 3 shows the test results with the choice of different dimensions n. We provide the minimum, maximum, and average number of iterations, as well as CPU time for 30 experiments of each n.

Dimension	Number of iterations			CPU time(s)		
	min	max	average	min	max	average
n=10	127	551	289.0333333	2.4492	10.5457	5.832343333
n=50	92	1023	318.5333333	3.1044	30.7478	9.806076667
n=100	65	1011	331.9333333	6.2244	101.2914	33.11329333

Table 3. Results on iterations and CPU time of algorithms with randomly generated data table3

5. Conclusions

sec:5

We proposed a subgradient-based method for nonconvex nonsmooth multiobjective optimization where the objective function is componentwise strongly quasiconvex. Under standard assumptions, we proved that the sequence generated by this method converges to an efficient solution of the problem, regardless of the initial point.

The numerical experiments presented in this paper are just for illustrative purposes. Our main motivation for studying the subgradient projection method is not its immediate practical value since the subgradient projection method is known to be "slow" even in the scalar case. However, we believe that this work could pave the way for the development of more efficient computational methods, such as bundle methods or cutting-plane algorithms for multiobjective optimization, as discussed in [10] in the convex setting. Additionally, it may contribute to the implementation of extragradient-type methods for vector equilibrium problems and vector variational inequalities, given that the extragradient method proposed in [22] has proven useful for scalar nonconvex equilibrium problems.

In a subsequent work, we will focus on inexact versions for subgradient methods in quasiconvex multiobjective optimization problems and its complexity analysis.

Disclosure statement

No potential conflict of interest was reported by the authors.

Availability of data and materials

No data were used to support this work. The MATLAB codes utilized are available from all authors upon reasonable request.

Funding

This research was partially supported by ANID–Chile under project Fondecyt Regular 1241040 (Lara), BASAL fund FB210005 for center of excellence from ANID-Chile (Marcavillaca) and the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.02 2023.01 (Thang).

Acknowledgements

We thanks both reviewers and the associated editor for their comments and suggestions which helped to improve the quality of the paper.

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