An Extragradient Projection Method for Strongly Quasiconvex Equilibrium Problems with Applications

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Abstract

We discuss an extragradient projection method for dealing with equilibrium problems which are strongly quasiconvex on its second argument. The algorithm combines a proximal step with a subgradient projection step using a generalized subdifferential, which is specially useful for dealing with this class of generalized convex functions, and also with a line search. As a consequence, the usual assumption regarding the relationship between the Lipschitz-type parameter and the modulus of strong quasiconvexity is not longer needed for ensuring the convergence of the generated sequence to the solution of the problem. Furthermore, numerical experiments for classes of nonconvex mixed variational inequalities based on fractional programming problems are given in order to show the performance of our proposed method.

Keywords: Nonconvex optimization; Equilibrium problems; Extragradient methods; Subgradient methods; Variational inequalities

1 Introduction

sec:1

Equilibrium problems (EP henceforth) are a very general framework for many important applied mathematical problems such as continuous optimization, variational inequalities, fixed point problems and Nash equilibrium problems among others. It can be formulated as follow:

$$\text{Find } \overline{x} \in K: \quad f(\overline{x}, y) \geq 0, \ \forall \ y \in K, \tag{1.1}$$

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where K is a nonempty closed convex set in $\mathbb{R}^n$, and $f : K \times K \to \overline{\mathbb{R}}$ is a bifunction such that $K \times K \subseteq \mathrm{dom}\, f$ and $f(x, x) = 0$ for any $x \in K$.

This formulation was first introduced, as far as we know, by Nikaido-Isoda in [28]. Later, it was named *equilibrium problem* in the famous paper of Blum and Oettli [6] and since then, it gained a lot of interest from both theoretical and algorithmic point of view (see [4, 5, 24, 36] and the references therein). The existence of solution for EP was proved by KyFan [22] under a quasiconvexity assumption on the bifunction with respect to the second argument and the compactness of K. Extension of this existence result for the nonconvex case in different directions may be found in [2, 3, 10, 15] and references therein.

With those existence results already obtained, in the last decades many researchers have put their efforts for developing algorithms for solving EP, specially in the convex case in which we can use the rich theory of convex analysis and convex optimization as a basic tool (see for example [4, 36, 37]). Algorithms for the convex case are very well developed and many accelerations/flexibilizations/generalizations have been implemented. In order to extend those interesting results to the nonconvex case, it is important to mention some difficulties, the main one is that several of the existing algorithms for EP were built based on the auxiliary principle (see [26]) which does not hold true for the nonconvex case (see, for instance, [39, page 12]).

Some efforts for dealing with the nonconvex case have been done in the very recent years, as the proximal point-type algorithm in [16] and the subgradient-type method in [38], both for the quasiconvex case. Those algorithms have some limitations; the proximal point-type algorithm requieres a relationship between the Lipschitz-type paramenter of f and the modulus of strong quasiconvexity of the second argument of f for ensuring convergence to the optimal solution while the subdgradient-type algorithm is slow and does not decrease at every step (as usual for subgradient-type algorithms). For dealing with this situations, the authors in [39] proposed an extragradient algorithm when the bifunction is quasiconvex in its second argument and they proved that the generated sequence converges to a quasi-solution of problem EP, while the authors in [17] proposed a two-step extragradient algorithm (an algorithm which computes two proximal step at every iteration) and they proved that their generated sequences converges to the solution of the problem faster than the usual proximal point-type algorithm, but they can not droped the assumption regarding the relationship between the Lipschitz-type paramenter of f and the modulus of strong quasiconvexity of the second argument of f

In this paper, we propose an extragradient projection method for solving EP in the strongly quasiconvex case. It consists of a proximal step, a projected subgradient step which uses the strong subdifferential in the subgradient part, specially designed for strongly quasiconvex functions [20], and also with a line search step. As a consequence, we ensures that the sequence generated by our algorithm converges to the unique solution of the EP without requiring any Lipschitz-type property of the function and, moreover, this convergence satisfies a type of linear convergence rate. Furthermore, we provide numerical illustrations of our theoretical results in classes of nonconvex mixed variational inequalities.

2 Preliminaries and Basic Definitions

sec:2

The inner product in $\mathbb{R}^n$ and the Euclidean norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The set $]0, +\infty[$ is denoted by $\mathbb{R}_{++}$. For a nonempty, convex and closed set $K \subseteq \mathbb{R}^n$ and an arbitrary $x \in \mathbb{R}^n$, there exists a unique element in K, denoted by $P_K(x)$ such that

$$\|u - P_K(u)\| \leq \|u - w\| \ \forall\, w \in K.$$

The operator $P_K : \mathbb{R}^n \to K$ is the *projection operator onto* K, and it satisfies that

$$v = P_K(u) \iff \langle u - v, w - v \rangle \leq 0, \ \forall\ w \in K. \tag{2.1}$$

proj_K

Given any extended-valued function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the effective domain of h is defined by $\operatorname{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. It is said that h is proper if $\operatorname{dom} h$ is nonempty and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$. The notion of properness is important when dealing with minimization problems.

It is indicated by $\operatorname{epi} h := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$ the epigraph of h, by $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ the sublevel set of h at the height $\lambda \in \mathbb{R}$ and by $\operatorname{argmin}_{\mathbb{R}^n} h$ the set of all minimal points of h. A function h is lower semicontinuous at $\overline{x} \in \mathbb{R}^n$ if for any sequence $\{x_k\}_k \in \mathbb{R}^n$ with $x_k \to \overline{x}$, $h(\overline{x}) \leq \liminf_{k \to +\infty} h(x_k)$. Furthermore, the current convention $\sup_{\emptyset} h := -\infty$ and $\inf_{\emptyset} h := +\infty$ is adopted.

A function h with convex domain is said to be

(*a*) convex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1 - \lambda) y) \leq \lambda h(x) + (1 - \lambda) h(y), \ \forall\ \lambda \in [0, 1], \tag{2.2}$$

def:convex

(*b*) strongly convex on $\operatorname{dom} h$ with modulus $\gamma \in]0, +\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0, 1]$, we have

$$h(\lambda y + (1 - \lambda) x) \leq \lambda h(y) + (1 - \lambda) h(x) - \lambda (1 - \lambda) \frac{\gamma}{2} \|x - y\|^2, \tag{2.3}$$

strong:convex

(*c*) semistrictly quasiconvex if, given any $x, y \in \operatorname{dom} h$, with $h(x) \neq h(y)$, then

$$h(\lambda x + (1 - \lambda) y) < \max\{h(x), h(y)\}, \ \forall\ \lambda \in]0, 1[, \tag{2.4}$$

(*d*) quasiconvex if, given any $x, y \in \operatorname{dom} h$, then

$$h(\lambda x + (1 - \lambda) y) \leq \max\{h(x), h(y)\}, \ \forall\ \lambda \in [0, 1], \tag{2.5}$$

def:qcx

(*e*) strongly quasiconvex on $\operatorname{dom} h$ with modulus $\gamma \in]0, +\infty[$ if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0, 1]$, we have

$$h(\lambda y + (1-\lambda)x) \leq \max\{h(y), h(x)\} - \lambda(1-\lambda)\frac{\gamma}{2}\|x - y\|^2. \tag{2.6}$$ strong:quasiconvex

It is said that h is strictly convex (resp. strictly quasiconvex) if the inequality in (2.2) (resp. (2.5)) is strict whenever $x \neq y$.

The relationship between all these notions is summarizing below (we denote quasiconvex by qcx):

$$\begin{array}{ccccccc}
\text{strongly convex} & \Longrightarrow & \text{strictly convex} & \Longrightarrow & \text{convex} & \Longrightarrow & \text{qcx} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{strongly qcx} & \Longrightarrow & \text{strictly qcx} & \Longrightarrow & \text{semistrictly qcx} & & \\
 & & \Downarrow & & & & \\
 & & \text{qcx} & & & &
\end{array}$$

If in addition the function is lsc, then functions in all previous classes are quasiconvex.

All the reverse statements do not hold in general. For instance, the Euclidean norm $h_1(x) = \|x\|$ is strongly quasiconvex without being strongly convex on any bounded convex set (see [19, Theorem 2]) and the function $h_2(x) = \frac{x}{1+|x|}$ is strictly quasiconvex without being strongly quasiconvex on $\mathbb{R}$ while the other counter examples are well-known (see [7, 11]).

The following existence result is the starting point of our research.

exist:unique **Lemma 2.1.** ([23, Corollary 3]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, lsc, and strongly quasiconvex function on $K \subseteq \operatorname{dom} h$ with modulus $\gamma > 0$. Then, $\operatorname{argmin}_K h$ is a singleton.*

Take $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper and $K \subseteq \mathbb{R}^n$ closed and convex such that $K \subseteq \operatorname{dom} h$. The *proximity operator of h on K of parameter $\beta > 0$ at $x \in \mathbb{R}^n$* is

$$\operatorname{Prox}_{\beta h} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \ \operatorname{Prox}_{\beta h}(K, x) = \operatorname*{argmin}_{y \in K} \left\{ h(y) + \frac{1}{2\beta}\|y - x\|^2 \right\}.$$

When $K = \mathbb{R}^n$, we write $\operatorname{Prox}_{\beta h}(\mathbb{R}^n, \cdot)$ simpler as $\operatorname{Prox}_{\beta h}(\cdot)$.

Given a proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the convex subdifferential of h at $\overline{x} \in \operatorname{dom} h$ is defined by

$$\partial h(\overline{x}) := \{\xi \in \mathbb{R}^n : \ h(y) \geq h(\overline{x}) + \langle \xi, y - \overline{x} \rangle, \ \forall \ y \in \mathbb{R}^n\}, \tag{2.7}$$ subd:usual

and by $\partial h(x) = \emptyset$ if $x \notin \operatorname{dom} h$.

The convex subdifferential is an outstanding tool in convex optimization, but when the function is nonconvex, the convex subdifferential is no longer useful. For this reason, several generalized subdifferentials for dealing with different classes of nonconvex functions have been introduced and studied (see for instance [32]). For the case of quasiconvex and strongly quasiconvex functions, we recall the following notion of generalized subdifferential proposed in [20].

The strong subdifferential of h at $\overline{x} \in \operatorname{dom} h \cap K$ is defined by

$$\partial^K_{\beta,\gamma} h(\overline{x}) := \{\xi \in \mathbb{R}^n : \max\{h(y), h(\overline{x})\} \geq h(\overline{x}) + \frac{\lambda}{\beta}\langle \xi, y - \overline{x}\rangle + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta} - \lambda\gamma\right)\|y - \overline{x}\|^2, \ \forall\, y \in K, \ \forall\, \lambda \in [0,1]\}. \tag{2.8}$$

strong:formula

Clearly, $\partial^K_{\beta,\gamma} h(\overline{x})$ is a closed and convex set. Moreover, we also have.

intdom

Lemma 2.2. ([20, Proposition 7(d)]) *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function, $\beta > 0$, $\gamma \geq 0$, $K \subseteq \mathbb{R}^n$, and $\overline{x} \in \operatorname{dom} h \cap K$. Then $\partial^K_{\beta,\gamma} h(\overline{x})$ is compact for every $\overline{x} \in \operatorname{int}(\operatorname{dom} h \cap K)$.*

A relationship between the strong subdifferential and the proximity operator of a strongly quasiconvex functions is given below.

pro:K

Lemma 2.3. ([20, Proposition 36]) *Let K be a closed and convex set in $\mathbb{R}^n$, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper and lsc function such that $K \subseteq \operatorname{dom} h$, $\beta > 0$ and $z \in K$. If h is strongly quasiconvex on K with modulus $\gamma \geq 0$, then*

$$\overline{x} \in \operatorname{Prox}_{\beta h}(K, z) \implies z - \overline{x} \in \partial^K_{\beta,\gamma} h(\overline{x}). \tag{2.9}$$

eq:sub

The strong subdifferential is nonempty for bigger classes of quasiconvex functions as we can see below.

subd:nonempty

Lemma 2.4. ([20, Corollary 38(a)]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper and lsc function such that $K \subseteq \operatorname{dom} h$, and $\beta > 0$. If h is strongly quasiconvex on K with modulus $\gamma > 0$, then $\partial^K_{\beta,\gamma} h(\overline{x}) \neq \emptyset$ for every $\overline{x} \in K$.*

A sufficient condition for ensuring the bounded of $\bigcup_{x \in K} \partial^K_{\beta,\gamma} h(x)$ is the folllowing.

bounded:subd

Lemma 2.5. ([25, Proposition 3.4]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, continuous and strongly quasiconvex function on $K \subseteq \operatorname{int} \operatorname{dom} h$ with modulus $\gamma > 0$. If K is compact, then $Y := \bigcup_{x \in K} \partial^K_{\beta,\gamma} h(x)$ is nonempty and bounded.*

Another useful property is the following.

pro:KLx

Lemma 2.6. ([20, Proposition 40]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper and lsc function such that $K \subseteq \operatorname{dom} h$, $\beta > 0$ and $\overline{x} \in K$. If h is strongly quasiconvex on K with modulus $\gamma > 0$, then*

$$y \in K \cap S_{h(\overline{x})}(h) \implies \langle \xi, y - \overline{x}\rangle \leq -\frac{\beta\gamma}{2}\|y - \overline{x}\|^2, \ \forall\, \xi \in \partial^K_{\beta,\gamma} h(\overline{x}). \tag{2.10}$$

A bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is monotone on $K \subseteq \mathbb{R}^n$, when

$$f(x,y) + f(y,x) \leq 0 \ \ \forall\, x, y \in K. \tag{2.11}$$

bif:mon

When the following weaker condition (see [11] for counter-examples for the opposite implication) is fulfilled

$$f(x,y) \geq 0 \implies f(y,x) \leq 0 \ \forall \ x, y \in K, \tag{2.12}$$ bif:pmon

it is said that f is pseudomonotone on K.

For a further study on generalized convexity and generalized monotonicity we refer to [7, 11, 19, 20, 23, 32, 34, 35] and references therein.

3 Extragradient Methods for Nonconvex Equilibrium Problems

sec:3

Let K be a closed and convex set in $\mathbb{R}^n$ and $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$. The equilibrium problem is defined by

$$\text{find } \overline{x} \in K : \quad f(\overline{x}, y) \geq 0, \ \forall \ y \in K. \tag{EP}$$ EP

Its solution set is denoted by $S(K, f)$.

A problem closely related to Problem (EP) is the Minty (or dual) equilibrium one that is defined as

$$\text{find } \overline{z} \in K : \quad f(y, \overline{z}) \leq 0, \ \forall \ y \in K. \tag{dEP}$$ DEP

Its solution set is denoted by $S_d(K, f)$.

rem:1 **Remark 3.1.** (*i*) *When we said that $f : K \times K \to \overline{\mathbb{R}}$ is a proper bifunction, we mean that $f(x, \cdot)$ is proper for every $x \in K$ and $f(\cdot, y)$ is proper for every $y \in K$.*

(*ii*) *Clearly, if f is pseudomonotone on K, then $S(K, f) \subseteq S_d(K, f)$. Conversely, if f is usc with respect to the first argument and convex with respect to the second argument, then $S_d(K, f) \subseteq S(K, f)$ (see [27] for instance). Furthermore, as was noted in [39, Lemma 1], $S_d(K, f) \subseteq S(K, f)$ still holds true when f is semistrictly quasiconvex with respect to the second argument instead of convex.*

We discuss next the assumptions on the equilibrium problem which will be used in the analysis of our algorithm.

($A0$) $f(x, x) = 0$ for all $x \in K$.

($A1$) f is continuous (jointly in both arguments) on an open set containing $K \times K$.

($A2$) f is pseudomonotone on K.

($A3$) For every $x \in K$, the function $f(x, \cdot)$ is strongly quasiconvex on K with modulus $\gamma > 0$.

rem:assumptions **Remark 3.2.** *We known that under assumption* (Ai) *with* $i = 0, 1, 2, 3$*, we have* $S_d(K, f) = S(K, f)$*, and by [16, Proposition 3.1],* $S(K, f) \neq \emptyset$ *is a singleton, i.e.,* $S_d(K, f) \neq \emptyset$ *is a singleton too by Remark 3.1(ii).*

The algorithm that we propose is based on the version presented in [39] (see also the algorithms in [4, Section 3.1]).

Algorithm 1 Proximal Extragradient Algorithm with linesearch for quasiconvex pseudomonotone EP's

EMSQ

Step 0. Take $x^1 \in K$, θ, $\varepsilon \in]0, 1[$, $k = 1$ and sequences $\{\alpha_k\}_k, \{\beta_k\}_k \subseteq \mathbb{R}_{++}$.

Step 1. Compute

$$y^k \in \mathrm{argmin}_{x \in K} \left(f(x^k, x) + \frac{1}{2\beta_k} \|x - x^k\|^2 \right) := \mathrm{Prox}_{\beta_k f(x^k, \cdot)}(K, x^k). \tag{3.1}$$

step:x

Step 2. (Armijo line-search rule) If $y^k = x^k$, then STOP, $\{x^k\} = S(K, f)$. Otherwise, find the smallest positive interger m such that

$$\begin{cases} f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\varepsilon}{2\beta_k} \|y^k - x^k\|^2, \\ \text{where } z^{k,m} := (1 - \theta^m) x^k + \theta^m y^k, \end{cases} \tag{3.2}$$

alg:ineq

and set $\theta_k = \theta^m$, $z^k = z^{k,m}$ and go to Step 3.

Step 3. Take $\xi^k \in (\partial^K_{\beta_k, \gamma})_2 f(z^k, x^k)$ and

$$x^{k+1} = P_K \left(x^k - \alpha_k \xi^k \right). \tag{3.3}$$

step:subd

Step 4. If $x^{k+1} = x^k$, then STOP, $\{z^k\} = S(K, f)$. Otherwise, take $k = k + 1$ and go to Step 1.

Before discussing the convergence analysis, we explain the motivation for introducing Algorithm 1.

Remark 3.3. *The main difference between Algorithm 1 and [39, Algorithm 31] is that we use the strong subdifferential (see relation* (2.8)*) instead of the star subdifferential (used in [39]). This difference is not just theoretical, it has algorithmic consequences as, for instance, the strong subdifferential is compact at every point of the interior of the effective domain of the function (Lemma 2.2) while the star subdifferential is always unbounded, an inconvenient property for algorithmic purposes. In [39], the subgradient is taken from the star subdifferential and it need to be normalized, thus the subgradient that they used belongs to the intersection of the star subdifferential and the unit ball in* $\mathbb{R}^n$*. In our case, Algorithm 1 can take any vector from the strong subdifferential. Furthemore, since the strong subdifferential was motivated by strongly quasiconvex* *functions, its geometrical structure will provide much more valuable information. We added two examples in order to show the difference between the strong subdifferential and the intersection of the star subdifferential and the unit ball. In these example, we can see the advantages of the strong subdifferential in contrast to the star subdifferential which is always unbounded because it is a cone.*

Let us recall the star subdifferential [30]:

$$\partial^{\star} h(x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle < 0, \ \forall \, y \in S^{<}_{h(x)}(h)\},$$

where $S^{<}_{h(x)}(h)$ *is the strict sublevel set of a function* h *at the height* $h(x)$ *with* $x \in \operatorname{dom} h$*. Now, let us consider the following examples:*

(*i*) *Let* $K = [-1, 1]$ *and* $h : \mathbb{R} \to \mathbb{R}$ *be given by* $h(x) = \sqrt{|x|}$*. By [23, Proposition 15],* h *is strongly quasiconvex on* K *with modulus* $\gamma_h = \frac{1}{2} > 0$*. Take* $\overline{x} = 0$ *and* $\beta = 1$*. Then, by [20, Remark 20] we have*

$$\partial^{[-1,1]}_{1, \frac{1}{2}} h(0) = \left[-\frac{3}{2}, \frac{3}{2}\right],$$

while $\partial^{\star} h(0) = \mathbb{R}$ *and* $\partial^{\star} h(0) \cap B(0, 1) = \{-1, 1\}$*.*

(*ii*) *Let* $h : \mathbb{R} \to \overline{\mathbb{R}}$ *be given by*

$$h(x) = \begin{cases} 0, & \text{if } x = 0, \\ -\frac{1}{x}, & \text{if } 0 < x \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that h *is strongly quasiconvex with modulus* $\gamma_h = 1$ *(see [18]). Take* $\overline{x} = 0$ *and* $\beta = 1$*. Then by [20, Remark 6(ii)] we have*

$$\partial^{\mathbb{R}}_{1,1} h(0) = \left]-\infty, -\frac{1}{2}\right],$$

while $\partial^{\star} h(0) =] - \infty, 0[$ *and* $\partial^{\star} h(0) \cap B(0, 1) = \{-1\}$*.*

We observe that, in both cases, the strong subdifferential is smaller than the star subdifferential. For instance, in case (*i*) *in which the function is continuous, the star subdifferential is the whole space (an unuseful information while the strong subdifferential is compact. Furthermore, in the difficult case* (*ii*) *in which the function is not even lower semicontinuous, both subdifferentials are unbounded, but the strong subdifferential is smaller and, moreover, it has a positive distance with the* 0*, a situation which could avoid the algorithm to stop in virtue of approximate errors.*

Remark 3.4. *We mention that relation* (3.1) *is just the iterative step of the standard PPA for the minimization of the function* $f^k(y) := f(x^k, y)$*, whose existence is guaranteed by assumptions* (*A*1)*,* (*A*3) *and Lemma 2.1. Furthermore, by assumption* (*A*3)*, for every* $x^k \in K$*, there exists* $\xi^k \in (\partial^K_{\beta_k, \gamma})_2 f(z^k, x^k)$ *by Lemma 2.4. Finally, the stopping citeria* $y^k = x^k$ *can be validated similarly to [16, Proposition 3.2].*

The following result is a variant of [39, Propositions 2 and 3] and proves that the line-search step (3.2) is well defined when $x^k \neq y^k$.

prop1 **Proposition 3.1.** *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers, and $\{x^k\}_k$, $\{y^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. If $y^k \neq x^k$, then the following assertions hold:*

- (a) *There exists $m \in \mathbb{N}$ such that relation (3.2) holds provided that f satifies (Ai) for $i = 0, 1$.*
- (b) *If f satisfies the assumptions (Ai) with $i = 0, 3$, then*

$$f(z^k, x^k) > 0. \tag{3.4}$$ great:0

- (c) $0 \notin (\partial^K_{\beta_k,\gamma})_2 f(z^k, x^k)$ *if f satisfies (Ai) with $i = 0, 1, 3$.*

Proof. (a) We proceed by contradiction. Suppose that for every $m > 0$, we have

$$f(z^{k,m}, x^k) - f(z^{k,m}, y^k) < \frac{\varepsilon}{2\beta_k}\|y^k - x^k\|^2.$$

Since $\lim_{m\to+\infty} z^{k,m} = x^k$ (see (3.2)), taking the $\liminf m \to +\infty$ on both sides of the inequality above and using $(A1)$, we have

$$f(x^k, x^k) - f(x^k, y^k) = \liminf_{m\to+\infty} \left(f(z^{k,m}, x^k) - f(z^{k,m}, y^k)\right) \leq \frac{\varepsilon}{2\beta_k}\|y^k - x^k\|^2,$$

which together with $(A0)$ implies that

$$0 \leq f(x^k, y^k) + \frac{\varepsilon}{2\beta_k}\|y^k - x^k\|^2. \tag{3.5}$$ eq:LU

On the other hand, it follows from (3.1) that

$$f(x^k, y^k) + \frac{1}{2\beta_k}\|y^k - x^k\|^2 \leq f(x^k, x) + \frac{1}{2\beta_k}\|x - x^k\|^2, \ \forall\, x \in K.$$

Putting $x = x^k$, thanks to $(A0)$, the latter inequality becomes

$$f(x^k, y^k) + \frac{1}{2\beta_k}\|y^k - x^k\|^2 \leq 0. \tag{3.6}$$ eq:LA

It follows from (3.5) and (3.6) that $\varepsilon \geq 1$, which contradicts the fact that $\varepsilon \in]0, 1[$.

(b) From (3.2) and $x^k \neq y^k$, we have $f(z^k, x^k) > f(z^k, y^k)$. Note that $z^k = (1 - \theta_k)x^k + \theta_k y^k$ with $\theta_k \in]0, 1[$. By the strong quasiconvexity of $f(z^k, \cdot)$ with modulus $\gamma > 0$ on K and (A0), we have

$$\begin{aligned}
0 = f(z^k, z^k) &\leq \max\{f(z^k, x^k), f(z^k, y^k)\} - \frac{\gamma}{2}\theta_k(1 - \theta_k)\|x^k - y^k\|^2, \\
&= f(z^k, x^k) - \frac{\gamma}{2}\theta_k(1 - \theta_k)\|x^k - y^k\|^2, \\
&< f(z^k, x^k).
\end{aligned}$$

(c) Follows from (b) and Lemma 2.6. □

Another stopping criteria for Algorithm 1 is given below.

Proposition 3.2. *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Suppose that assumptions $(A3)$ hold. If $x^{k+1} = x^k$, then z^k is a solution of* (EP).

Proof. If $x^{k+1} = x^k$, then by relation (2.1) we have

$$\langle \xi^k, y - x^k \rangle \geq 0, \ \forall \, y \in K.$$

Since $\xi^k \in (\partial^K_{\gamma, \beta_k})_2 f(z^k, x^k)$, it follows that

$$\begin{aligned}\max\{f(z^k, y), f(z^k, x^k)\} &\geq f(z^k, x^k) + \frac{\lambda}{\beta_k}\langle \xi^k, y - x^k \rangle + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta_k} - \lambda\gamma\right)\|y - x^k\|^2, \\ &\geq f(z^k, x^k) + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta_k} - \lambda\gamma\right)\|y - x^k\|^2, \ \forall \, y \in K, \ \forall \, \lambda \in [0, 1].\end{aligned}$$

By taking $0 < \lambda < \frac{\gamma\beta_k}{1+\gamma\beta_k}$, we have

$$\begin{aligned}&\max\{f(z^k, y), f(z^k, x^k)\} > f(z^k, x^k), \ \forall \, y \in K \\ &\iff f(z^k, y) > f(z^k, x^k), \ \forall \, y \in K\end{aligned}$$

i.e., $\{z^k\} = S(K, f)$ by relation (3.4). □

For the next result, we simple repeat the proof of [25, Proposition 3.1] to $(\partial^K_{\beta,\gamma})_2 f(z^k, x^k)$, so the proof is omitted.

`obvious` **Proposition 3.3.** ([25, Proposition 3.1]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \mathbb{R}$ be a bifunction, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Then, the following statements hold:*

(a) $\|x^{k+1} - x^k\| \leq \alpha_k \|\xi^k\|$.

(b) For every $w \in K$, we have

$$\|x^{k+1} - w\|^2 \leq \|x^k - w\|^2 + (\alpha_k)^2\|\xi^k\|^2 + 2\alpha_k\langle \xi^k, w - x^k \rangle. \tag{3.7}$$ `RM1`

Before analyzing the convergence of Algorithm 1 we show a key result.

`key` **Proposition 3.4.** *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Suppose that assumptions (Ai) with $i = 0, 3$ holds, then for any $x^* \in S_d(f, K)$ we have*

$$\|x^{k+1} - x^*\|^2 \leq (1 - \gamma\alpha_k\beta_k)\|x^k - x^*\|^2 + (\alpha_k)^2\|\xi^k\|^2. \tag{3.8}$$ `eq:02`

Proof. Let $x^* \in S_d(K,f)$. Recall that (3.7) with $w = x^*$ gives

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 + (\alpha_k)^2\|\xi^k\|^2 + 2\alpha_k\langle \xi^k, x^* - x^k\rangle. \tag{3.9}$$ RM2

Now, since $\xi^k \in (\partial^K_{\gamma,\beta_k})_2 f(z^k, x^k)$, we have

$$\max\{f(z^k,y), f(z^k,x^k)\} \ge f(z^k,x^k) + \frac{\lambda}{2}\left(\gamma - \frac{\lambda}{\beta_k} - \lambda\gamma\right)\|y - x^k\|^2 + \frac{\lambda}{\beta_k}\langle \xi^k, y - x^k\rangle,\ \forall\, \lambda \in [0,1],\ \forall\, y \in K. \tag{3.10}$$

Take $y = x^* \in S_d(K,f)$, thus $f(z^k,x^*) \le 0$. Since $0 < f(z^k,x^k)$ by relation (3.4), we have $f(z^k,x^*) < f(z^k,x^k)$, i.e.,

$$\langle \xi^k, x^* - x^k\rangle \le -\frac{\beta_k}{2}\left(\gamma - \frac{\lambda}{\beta_k} - \lambda\gamma\right)\|x^* - x^k\|^2,\ \forall\, \lambda \in]0,1]. \tag{3.11}$$ RM3

Letting $\lambda \to 0$ in (3.11) we deduce

$$\langle \xi^k, x^* - x^k\rangle \le -\frac{\gamma\beta_k}{2}\|x^* - x^k\|^2,\ \forall\, k \in \mathbb{N}. \tag{3.12}$$ RM4

The desired conclusion in (3.8) follows from (3.9) and (3.12). □

In order to establish the convergence of Algorithm 1 to the solution of (EP), we consider the following compatibility conditions:

$(C1)$ $K \subseteq \operatorname{int}\operatorname{dom} f(x,\cdot)$ for any $x \in K$.

$(C2)$ There exists $M > 0$ such that for every $x \in K$ and every $k \in \mathbb{N}$, we have $(\partial^K_{\beta_k,\gamma})_2 f(x,x^k) \subseteq \mathbb{B}(0,M)$.

$(C3)$ The sequence $\{\alpha_k\}_k \subseteq]0,+\infty[$ is such that $0 < \alpha_k < \frac{1}{\gamma\beta'}$ for every $k \in \mathbb{N}$, with $\beta' := \inf_k \beta_k > 0$, and

$$\sum_{k=0}^{\infty} \alpha_k = +\infty, \quad \sum_{k=1}^{\infty} \alpha_k^2 < +\infty. \tag{3.13}$$

Remark 3.5. *Assumptions (Ci) with $i = 1,2,3$ are usual assumptions for subgradient algorithms. Indeed, $(C1)$ and $(C2)$ are actually used even for convex functions with the convex subdifferential (see Assumption 8.7(C) and Assumption 8.12 in [1], respectively). Moreover, as in the convex case, assumption $(C2)$ is not too restrictive in virtue of Lemmas 2.2 and 2.5 which are the analogous properties for the strong subdifferential with strongly quasiconvex functions.*

A useful result for the convergence of Algorithm 1 is given below. The particular case of the minimization problem may be found in [25, Proposition 3.5].

limit **Proposition 3.5.** *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Suppose that assumption (Ai) with $i = 0, 1, 3$ and (Ci) with $i = 1, 2, 3$ holds, then*

(*a*) *for every $x^* \in S_d(K, f)$ we have*

$$\lim_{k\to\infty} \langle \xi^k, x^k - x^* \rangle = 0. \qquad (3.14)$$ eq:limit

(*b*) $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < +\infty$. *In particular,* $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

Proof. (a): By Proposition 3.3(b), we have for $x^* \in S_d(K, f)$ that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + (\alpha_k)^2\|\xi^k\|^2 + 2\alpha_k\langle \xi^k, x^* - x^k\rangle. \qquad (3.15)$$ L3

Since $\xi^k \in (\partial^K_{\beta_k,\gamma})_2 f(z^k, x^k)$ we have from (3.12) (into the proof of Proposition 3.4) that

$$\langle \xi^k, x^* - x^k\rangle \leq -\frac{\gamma\beta_k}{2}\|x^* - x^k\|^2 \leq 0 \iff \langle \xi^k, x^k - x^*\rangle \geq 0, \ \forall\, k \in \mathbb{N}.$$

Hencer, it follows from relation (3.15) and assumption $(C2)$ that

$$0 \leq \sum_{k=0}^{N} \alpha_k\langle \xi^k, x^k - x^*\rangle \leq \frac{1}{2}\|x^0 - x^*\|^2 - \frac{1}{2}\|x^{N+1} - x^*\|^2 + \frac{M}{2}\sum_{k=0}^{N}\alpha_k^2.$$

This implies that $\sum_{k=0}^{\infty} \alpha_k\langle \xi^k, x^k - x^*\rangle < +\infty$ because $\sum_{k=0}^{\infty} \alpha_k^2 < +\infty$ by $(C3)$. Moreover, since $\sum_{k=0}^{\infty} \alpha_k = +\infty$ also by $(C3)$, we conclude that $\lim_{k\to\infty}\langle \xi^k, x^k - x^*\rangle = 0$.

(b): It follows directly by summing up in Proposition 3.3(a) (after squaring both sides) and taking into account (Ci) with $i = 1, 2, 3$, i.e.,

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 \leq M^2 \sum_{k=0}^{\infty} \alpha_k^2 < +\infty,$$

which implies, in particular, that $\lim_{k\to+\infty} \|x^{k+1} - x^k\|^2 = 0$. □

In the following result, we prove that the sequences $\{x^k\}_k$, generated by Algorithm 1, is bounded under assumptions (Ci) with $i = 1, 2, 3$ and converges to the unique solution of problem (EP).

theo:main **Theorem 3.1.** *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x, \cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$, $\{y^k\}_k$, $\{z^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Suppose that assumptions (Ai) with $i = 0, 1, 2, 3$ and (Cj) with $j = 1, 2, 3$ holds. Then the following assertions holds:*

(a) *For* $x^* \in S_d(K,f)$*, the limit* $\lim_{k\to\infty} ||x^k - x^*|| = 0$ *and*

$$\lim_{k\to+\infty} \|z^k - x^*\| = \lim_{k\to+\infty} \theta_k \|y^k - x^*\|, \tag{3.16}$$ eq:16

exists.

(b) *The sequences* $\{x^k\}_k$*,* $\{y^k\}_k$ *and* $\{z^k\}_k$ *are bounded and* $\{x^k\}_k$ *converges to* $x^* = S(K,f)$*.*

Proof. (a): By assumption $(C2)$, we have $\|\xi^k\| \leq M$ for every $k \in \mathbb{N}$. Hence, for every $x^* \in \Omega$ we have from (3.8) (with $w = x^*$) that

$$\gamma\beta_k\alpha_k\|x^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + M^2\alpha_k^2,$$

and, since $\beta' = \inf_k \beta_k$, we obtain from the above inequality that,

$$\begin{aligned}\gamma\beta' \sum_{k=0}^{N} \alpha_k\|x^k - x^*\|^2 &\leq \|x^0 - x^*\|^2 - \|x^{N+1} - x^*\|^2 + M^2\sum_{k=0}^{N}\alpha_k^2 \\ &\leq \|x^0 - x^*\|^2 + M^2\sum_{k=0}^{N}\alpha_k^2.\end{aligned}$$

Letting $N \to +\infty$ taking into account that $\sum_{k=0}^{\infty}\alpha_k^2 < +\infty$ by $(C3)$, we deduce that $\sum_{k=0}^{+\infty}\alpha_k\|x^k - x^*\|^2 < +\infty$, consequently, due to $\sum_{k=0}^{\infty}\alpha_k = \infty$ by $(C3)$, we conclude that $\lim_{k\to+\infty}\|x^k - x^*\|^2 = 0$. As a consequence, $\{x^k\}_k$ is bounded, i.e., it has cluster points. The limit in (3.16) follows from the fact that $z^k - x^k = \theta_k(y^k - x^k)$ and $\lim_{k\to+\infty}\|x^k - x^*\| = 0$. Therefore, $\{x^k\}_k$ is bounded and converges to $x^* = S_d(K,f) = S(K,f)$, where the last equality follows from Remark 3.2.

(b): Since for every $k \in \mathbb{N}$, $y^k \in \mathrm{Prox}_{\beta_k f(x^k,\cdot)}(x^k)$, it holds by Lemma 2.3 that $x^k - y^k \in (\partial^K_{\beta_k,\gamma})_2 f(x^k, y^k)$, then $\{x^k - y^k\}_k$ is bounded by $(C2)$, and hence $\{y^k\}_k$ is also bounded (because $\{x^k\}_k$ is bounded). Then $\{z^k\}_k$ is bounded too by (3.16). □

In the following result, we prove that the sequences $\{y^k\}_k$ and $\{z^k\}_k$ also converge to the unique solution of the problem (EP).

prop:tech **Proposition 3.6.** *Let* $K \subseteq \mathbb{R}^n$ *be a closed and convex set,* $f : K \times K \to \overline{\mathbb{R}}$ *be a proper bifunction such that* $K \subseteq \mathrm{dom}\, f(x,\cdot)$ *for every* $x \in K$*,* $\{\alpha_k\}_k$ *be a sequence of positive numbers and* $\{x^k\}_k$*,* $\{y^k\}_k$*,* $\{z^k\}_k$ *and* $\{\xi^k\}_k$ *be the sequences generated by Algorithm 1. Suppose that assumptions* (Ai) *with* $i = 0,1,2,3$ *and* (Cj) *with* $j = 1,2,3$ *holds. Let* $x^* = S_d(K,f)$*, and let* $\widehat{y}$ *and* $\widehat{z}$ *be any cluster points of* $\{y^k\}_k$ *and* $\{z^k\}_k$*, respectively. Then the following assertions hold:*

(a) $f(\widehat{z}, x^*) = 0$*;*

(b) $x^* = \widehat{y}$*.*

As a consequence, the sequences $\{y^k\}_k$ *and* $\{z^k\}_k$*, generated converges also to* $x^* = S(K,f)$.

Proof. (a): It follows from Theorem 3.1 that the sequences $\{y^k\}_k$ and $\{z^k\}_k$ are bounded, i.e., they have cluster points. Let $\widehat{y}$ and $\widehat{z}$ be any cluster points of $\{y^k\}_k$ and $\{z^k\}_k$, respectively, then there are subsequences $\{y^{k_\ell}\}_\ell \subseteq \{y^k\}_k$ and $\{z^{k_\ell}\}_\ell \subseteq \{z^k\}_k$ such that

$$\lim_{\ell\to+\infty} y^{k_\ell} = \widehat{y} \quad \text{and} \quad \lim_{\ell\to+\infty} z^{k_\ell} = \widehat{z}. \tag{3.17}$$

subseq

Since $x^k \to x^* \in S_d(K,f)$, as $k \to +\infty$, it follows from Proposition 3.5(a) that

$$\lim_{\ell\to+\infty} \langle \xi^{k_\ell}, x^{k_\ell} - x^* \rangle = \lim_{k\to+\infty} \langle \xi^k, x^k - x^* \rangle = 0, \tag{3.18}$$

limit:0

where $\{x^{k_\ell}\}_\ell$ is any subsequence of $\{x^k\}_k$.

Since, for every $k \in \mathbb{N}$, $f(z^k, x^k) > 0$ by relation (3.4), it follows from (3.17) and $(A1)$ that $f(\widehat{z}, x^*) = \lim_{k\to+\infty} f(z^{k_\ell}, x^{k_\ell}) \geq 0$, i.e., $f(\widehat{z}, x^*) \geq 0$.

To show the contrary inequality, we deal by contradiction. In fact, assume that $f(\widehat{z}, x^*) > 0$, i.e., there exists $a > 0$ such that $f(\widehat{z}, x^*) \geq a$. Since $f(\cdot,\cdot)$ is continuous, there exists $\ell_0 \in \mathbb{N}$ such that

$$f(z^{k_\ell}, x^{k_\ell}) > \frac{a}{2}, \ \forall\, \ell \geq \ell_0. \tag{3.19}$$

eq:TA

Since $x^* \in S_d(K,f)$, we have $f(\widehat{z}, x^*) \leq 0$. Again using the continuity of $f(\cdot,\cdot)$, there exist $\epsilon_1, \epsilon_2 > 0$ such that, for any $z \in B(\widehat{z}, \epsilon_1)$ and $y \in B(x^*, \epsilon_2)$, we have

$$f(z,y) \leq \frac{a}{2}.$$

On the other hand, $\lim_{\ell\to+\infty} z^{k_\ell} = \widehat{z}$ implies that there exists $\ell_1 \in \mathbb{N}$ such that, for any $\ell \geq \ell_1$, $z^{k_\ell} \in B(\widehat{z}, \epsilon_1)$. So, for $\ell \geq \max\{\ell_0, \ell_1\}$ and $y \in B(x^*, \epsilon_2)$, we have

$$f(z^{k_\ell}, y) \leq \frac{a}{2} < f(z^{k_\ell}, x^{k_\ell}).$$

This implies that $y \in S_{f(z^{k_\ell}, x^{k_\ell})}(f(z^{k_\ell}, \cdot))$, and since $\xi^{k_\ell} \in (\partial^K_{\beta_{k_\ell},\gamma})_2 f(z^{k_\ell}, x^{k_\ell})$, it follows from Lemma 2.6 that

$$\langle \xi^{k_\ell}, y - x^{k_\ell} \rangle \leq -\frac{\gamma\beta}{2} \|y - x^{k_\ell}\|^2 \leq 0.$$

Taking $y = x^* + \epsilon_2 \frac{\xi^{k_\ell}}{\|\xi^{k_\ell}\|^2}$ ($\|\xi^{k_\ell}\| \neq 0$ by Proposition 3.1(c)), we have

$$\left\langle \xi^{k_\ell}, x^* + \epsilon_2 \frac{\xi^{k_\ell}}{\|\xi^{k_\ell}\|^2} - x^{k_\ell} \right\rangle \leq 0 \iff \langle \xi^{k_\ell}, x^{k_\ell} - x^* \rangle \geq \epsilon_2 > 0,$$

which contradicts relation (3.18). Hence $f(\widehat{z}, x^*) = 0$.

(b): First we recall from (3.2) that $f(z^{k_\ell}, x^{k_\ell}) \geq f(z^{k_\ell}, y^{k_\ell})$, for every $\ell \in \mathbb{N}$. Then letting $\ell \to +\infty$ and using the continuity of $f(\cdot,\cdot)$ and part (a), we deduce that $f(\widehat{z}, \widehat{y}) \leq f(\widehat{z}, x^*) = 0$.

Furthermore, the sequence $\{\theta_k\}_k \subseteq]0,1[$, i.e., it has clusted points. Let $\widehat{\theta}$ be any cluster point of $\{\theta_k\}_k$, thus there exists a subsequence $\{\theta_{k_\ell}\}_\ell$ such that $\theta_{k_\ell} \to \widehat{\theta}$ as $\ell \to +\infty$.

Taking the limit as $\ell \to +\infty$ on the subsequence k_ℓ in (3.2), using the limit in (3.17), we deduce that $\widehat{z} = (1-\widehat{\theta})x^* + \widehat{\theta}\widehat{y}$. It yields from $(A4)$ that

$$0 = f(\widehat{z},\widehat{z}) \leq \max\{f(\widehat{z},x^*), f(\widehat{z},\widehat{y})\} - \widehat{\theta}(1-\widehat{\theta})\frac{\gamma}{2}\|x^* - \widehat{y}\|^2. \tag{3.20}$$

Since $f(\widehat{z},x^*) = 0$ by part (a) and $f(\widehat{z},\widehat{y}) \leq 0$, we have

$$\widehat{\theta}(1-\widehat{\theta})\frac{\gamma}{2}\|x^* - \widehat{y}\|^2 \leq \max\{0, f(\widehat{z},\widehat{y})\} = 0 \iff x^* = \widehat{y}.$$

As a consequence, $\{y^k\}_k$ converges to $x^* = S(K,f)$ and also $\{z^k\}_k$ converges to $x^* = S(K,f)$ by relation (3.16), which completes the proof. □

As a consequence of Proposition 3.4 and Theorem 3.1 we also have a convergence results for Algorithm 4.1, a kind of linear convergence rate.

lintype:convrate

Corollary 3.1. *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $f : K \times K \to \overline{\mathbb{R}}$ be a proper bifunction such that $K \subseteq \operatorname{dom} f(x,\cdot)$ for every $x \in K$, $\{\alpha_k\}_k$ be a sequence of positive numbers and $\{x^k\}_k$, $\{y^k\}_k$, $\{z^k\}_k$ and $\{\xi^k\}_k$ be the sequences generated by Algorithm 1. Suppose that assumptions (Ai) with $i = 0,1,2,3$ and (Cj) with $j = 1,2,3$ holds. Then for every $k \in \mathbb{N}$, we have*

$$\|x^{k+1} - x^*\|^2 \leq \Pi_{j=0}^{k}\left(1 - \gamma\beta'\alpha_j\right)\|x^0 - x^*\|^2 + M^2\sum_{j=1}^{k}\alpha_j^2. \tag{3.21}$$

eq:LC

Proof. From (3.8) and assumption $(C2)$ it follows for any $k \in \mathbb{N}$ that

$$\|x^{k+1} - \overline{x}\|^2 \leq (1 - \gamma\beta_k\alpha_k)\|x^k - \overline{x}\|^2 + M^2\alpha_k^2.$$

Thus, since $0 < \alpha_k < \frac{1}{\gamma\beta'}$ for every $k \in \mathbb{N}$ with $\beta' := \inf_k \beta_k > 0$ by $(C3)$, we obtain recursively

$$\begin{aligned}\|x^{k+1} - \overline{x}\|^2 &\leq (1-\gamma\beta'\alpha_k)\left[(1-\gamma\beta'\alpha_{k-1})\|x^k - \overline{x}\|^2 + M^2\alpha_{k-1}^2\right] + M^2\alpha_k^2\\ &\ \ \vdots\\ &\leq \Pi_{j=0}^{k}\left(1-\gamma\beta'\alpha_j\right)\|x^0 - \overline{x}\|^2 + M^2\sum_{j=0}^{k}\alpha_j^2,\end{aligned}$$

and the result follows. □

4 Applications and Numerical Experiments

sec:4

In this section, we present applications in continuous optimization problems based on fractional programming.

4.1 Fractional Programming

Given a subset $K \subseteq \mathbb{R}^n$, and functions $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, we define the fractional minimization problem by

$$\min_{x \in K} \varphi(x) \; = \; \min_{x \in K} \frac{h(x)}{g(x)}. \tag{FMP}$$ FMP

This problem has been deeply studied in the literature (see [7, 14, 34, 35] among others) due to its concrete applications in several fields of mathematical sciences, especially in economics as, for instance, in theory of productivity as maximization of return/risk or profit/cost and minimization of cost/time among others.

It is important to mention that, in general, the problem (FMP) is not convex. Indeed, if, for instance, h is convex and g is affine, then φ is semistrictly quasiconvex, while if h is non-negative and convex, and g is positive and concave, then φ is semistrictly quasiconvex too, by [7, Theorem 2.3.8].

In the case when h is strongly convex, sufficient conditions for φ being strongly quasiconvex are given below.

prop:frac **Proposition 4.1.** ([17]) *Suppose that* $\varphi(x) = \frac{h(x)}{g(x)}$ *for all* $x \in \operatorname{dom} \varphi$ *with* $\operatorname{dom} \varphi$ *a convex set,* h *is strongly convex with modulus* $\gamma > 0$*,* g *is finite, positive and bounded from above by* M *on* $\operatorname{dom} \varphi$*. If any of the following conditions holds:*

- (a) *g is affine,*
- (b) *h is nonnegative on* $\operatorname{dom} \varphi$ *and g is concave,*
- (c) *h is nonpositive on* $\operatorname{dom} \varphi$ *and g is convex,*

then φ *is strongly quasiconvex with modulus* $\gamma' := \frac{\gamma}{M} > 0$.

The folllowing corollary is a direct consequence of Proposition 4.1.

quad:frac **Corollary 4.1.** ([17]) *Let* $A, B \in \mathbb{R}^n$ *be two matrices,* $a, b \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$, $h(x) = \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha$ *and* $g(x) = \frac{1}{2}\langle Bx, x\rangle + \langle b, x\rangle + \beta$*. Take* $K = \{x \in \mathbb{R}^n : \; m \leq g(x) \leq M\}$*, with* $0 < m < M$*, and* $\varphi(x) = \frac{h(x)}{g(x)}$*, with* $x \in K$*. Suppose that A is a positive definite matrix. If any of the following conditions holds:*

- (a) $B = 0$ *(the null matrix),*
- (b) *h is nonnegative on K and B is negative semidefinite,*
- (c) *h is nonpositive on K and B is positive semidefinite,*

then φ *is strongly quasiconvex with modulus* $\gamma' := \frac{\lambda_{\min}(A)}{M} > 0$.

In the next result, we propose a relationship between the convex and strong subdifferentials for classes of fractional functions.

prop:frac2 **Proposition 4.2.** *Suppose that* $\varphi(x) = \frac{h(x)}{g(x)}$ *for all* $x \in \operatorname{dom}\varphi$*, where* h *is strongly convex with modulus* $\gamma > 0$*,* g *is finite, affine, positive and bounded from above by* $M > 0$ *on* $\operatorname{dom}\varphi$*,* $\operatorname{dom}\varphi$ *is convex. Then for any* $\overline{x} \in \operatorname{dom}\varphi$*, we have*

$$\frac{\beta}{M}\partial(h - \alpha g)(\overline{x}) \subseteq \partial^{\operatorname{dom}\varphi}_{\beta,\gamma'}\varphi(\overline{x}),$$

with $\alpha := \varphi(\overline{x})$*.*

Proof. Take $y, \overline{x} \in \operatorname{dom}\varphi$. Then it follows from [17, Propsition 4.2] that for every $\lambda \in [0,1]$ we have

$$\varphi(\lambda\overline{x} + (1-\lambda)y) \leq \max\{\varphi(\overline{x}), \varphi(y)\} - \frac{\gamma}{2}\lambda(1-\lambda)\frac{1}{(\lambda g(\overline{x}) + (1-\lambda)g(y))}\|y - \overline{x}\|^2. \tag{4.1}$$

eq::fracstrong

Let $\alpha := \varphi(\overline{x})$ and $\xi \in \partial(h - \alpha g)(\overline{x})$. Then,

$$(h - \alpha g)(z) \geq \langle \xi, z - \overline{x}\rangle, \ \forall\ z \in \operatorname{dom}\varphi.$$

For $y \in \operatorname{dom}\varphi$ and $\lambda \in [0,1]$, we take $z = \lambda y + (1-\lambda)\overline{x}$, thus

$$(h - \alpha g)(\lambda y + (1-\lambda)\overline{x}) \geq \lambda\langle \xi, y - \overline{x}\rangle.$$

Since g is affine and positive, $g(\lambda y + (1-\lambda)\overline{x}) = \lambda g(y) + (1-\lambda)g(\overline{x}) > 0$, we can divide both sides by $\lambda g(y) + (1-\lambda)g(\overline{x})$, thus

$$\varphi(\lambda y + (1-\lambda)\overline{x}) - \varphi(\overline{x}) \geq \frac{\lambda}{\lambda g(y) + (1-\lambda)g(\overline{x})}\langle \xi, y - \overline{x}\rangle. \tag{4.2}$$

eq:Y0

Now, from relations (4.1) and (4.2), we obtain

$$\max\{\varphi(y), \varphi(\overline{x})\} - \varphi(\overline{x}) \geq \frac{1}{\lambda g(y) + (1-\lambda)g(\overline{x})}\big(\lambda\langle \xi, y - \overline{x}\rangle + \lambda(1-\lambda)\frac{\gamma}{2}\|x - y\|^2\big). \tag{4.3}$$

eq:Y1

Note that (4.3) holds for every $\lambda \in [0,1]$ and that

$$\max_{\lambda\in[0,1]}\big(\lambda\langle \xi, y - \overline{x}\rangle + \lambda(1-\lambda)\frac{\gamma}{2}\|x - y\|^2\big) \geq 0.$$

Furthermore, since $0 < \lambda g(y) + (1-\lambda)g(\overline{x}) < M$, we have

$$\begin{aligned}&\frac{1}{\lambda g(y) + (1-\lambda)g(\overline{x})}\max_{\lambda\in[0,1]}\big(\lambda\langle \xi, y - \overline{x}\rangle + \lambda(1-\lambda)\frac{\gamma}{2}\|\overline{x} - y\|^2\big)\\ &\geq \frac{1}{M}\max_{\lambda\in[0,1]}\big(\lambda\langle \xi, y - \overline{x}\rangle + \lambda(1-\lambda)\frac{\gamma}{2}\|\overline{x} - y\|^2\big).\end{aligned}$$

Then, for every $\lambda \in [0,1]$ and every $y \in \operatorname{dom}\varphi$, we have

$$\max\{\varphi(y), \varphi(\overline{x})\} \geq \varphi(\overline{x}) + \frac{1}{M}\big(\lambda\langle \xi, y - \overline{x}\rangle + \lambda(1-\lambda)\frac{\gamma}{2}\|\overline{x} - y\|^2\big).$$

Therefore, $\frac{\beta}{M}\xi \in \partial^{\operatorname{dom}\varphi}_{\beta,\gamma}\varphi(\overline{x})$. □

In order to provide more concrete examples. Let us consider $K \subseteq \mathbb{R}^n$ be a closed and convex set, $T, F : K \to \mathbb{R}^n$ be two operators and $G : K \to \mathbb{R}$ be a real-valued function. Then the Inverse Mixed Variational Inequality (IMVI) problem is given by:

$$\text{Find } x^* \in K : \quad \langle T(x^*), F(y) - F(x^*)\rangle + G(y) - G(x^*) \geq 0, \ \forall\, y \in K. \quad \text{(IMVI)}$$

IMVI

Problem (IMVI) is a highly general formulation, it includes several optimization problems such as inverse variational inequalities (see [12, 13, 40]), mixed variational inequalities (see [8, 9, 29]) and minimization problems among others. Moreover, we point out that the extended linear-quadratic programming studied in [33] can be formulated as an (IMVI) too (see [12]).

In the folllowing example, we show families of IMVI problems which satisfy our assumptions (Ai) with $i = 0, 1, 2, 3$.

ex:IMVI

Example 4.1. *Let $A, E_1, E_2 \in \mathbb{R}^{n\times n}$ be matrices, $b, c, g_1, g_2 \in \mathbb{R}^n$ and $d, h_1, h_2 \in \mathbb{R}$. Let us consider $K := \{x \in \mathbb{R}^n : \ M_1 \leq \langle c, x\rangle + d \leq M_2\}$ with $0 < M_1 < M_2 < +\infty$, $T : K \to K$, $F : K \to \mathbb{R}^n$ and $G : K \to \mathbb{R}$ be the operators and function defined by*

$$T(x) := x; \quad F(x) := \frac{Ax + b}{\langle c, y\rangle + d}, \quad G_1(x) := \frac{\frac{1}{2}\langle E_1 x, x\rangle + \langle g_1, x\rangle + h_1}{\langle c, y\rangle + d}. \quad (4.4)$$

ex:oper

By taking $f_1 : K \times K \to \mathbb{R}$ given by

$$f_1(x, y) := \langle x, F(y) - F(x)\rangle + G_1(y) - G_1(x), \quad (4.5)$$

equi:bifunction1

problem (IMVI) *reduces to an equilibrium problem.*

Let us check that bifunction f_1 defined in (4.5) *satisfies the assumptions (Ai) with $i = 0, 1, 2, 3$ under mild assumptions. Indeed, $(A0)$ and $(A1)$ are straightforward. Furthermore, if F is monotone, then assumption $(A2)$ holds.*

For $(A3)$: Note that f_1 can be rewritten as

$$\begin{aligned} f_1(x, y) &= \langle x, F(y) - F(x)\rangle + G_1(y) - G_1(x) \\ &= \frac{\langle Ay + b, x\rangle}{\langle c, y\rangle + d} + \frac{\frac{1}{2}\langle E_1 y, y\rangle + \langle g_1, y\rangle + h_1}{\langle c, y\rangle + d} - \frac{\langle Ax + b, x\rangle}{\langle c, x\rangle + d} - \frac{\frac{1}{2}\langle E_1 x, x\rangle + \langle g_1, x\rangle + h_1}{\langle c, x\rangle + d} \\ &= \frac{\frac{1}{2}\langle E_1 y, y\rangle + \langle A^T x + g_1, y\rangle + \langle b, x\rangle + h_1}{\langle c, y\rangle + d} - \frac{\langle Ax + b, x\rangle}{\langle c, x\rangle + d} - \frac{\frac{1}{2}\langle E_1 x, x\rangle + \langle g_1, x\rangle + h_1}{\langle c, x\rangle + d}. \end{aligned}$$

Hence, in virtue of Proposition 4.1, for any $x \in K$ the function $f_1(x, \cdot)$ is strongly quasiconvex on K when the matrix E_1 is positive definite, and its modulus of strongly quasiconvexity is $\gamma_1 = \frac{\lambda_{\min}(E_1)}{M} > 0$.

Now, if we consider

$$G_2(x) := \max\left\{\frac{\frac{1}{2}\langle E_1 x, x\rangle + \langle g_1, x\rangle + h_1}{\langle c, x\rangle + d}, \frac{\frac{1}{2}\langle E_2 x, x\rangle + \langle g_2, x\rangle + h_2}{\langle c, x\rangle + d}\right\}. \quad (4.6)$$

G:2

Then we define $f_2 : K \times K \to \mathbb{R}$ *by*

$$f_2(x, y) := \langle x, F(y) - F(x)\rangle + G_2(y) - G_2(x). \tag{4.7}$$ equi:bifunction2

Note that f_2 *is strongly quasiconvex on its second argument on* K *when both matrices* E_1, E_2 *are positive definite, and its modulus of strong quasiconvexity is* $\gamma_2 := \min\{\frac{\lambda_{min}(E_1)}{M_2}, \frac{\lambda_{min}(E_2)}{M_2}\}$.

Remark 4.1. *Note that in contrast to [16, Algorithm 1] and [17, Algorithm 1], our Algorithm 1 does not need that* f *satisfied the Lipschitz type condition:*

$$f(x, y) + f(y, z) \geq f(x, z) - \eta\big(\|x - y\|^2 + \|y - z\|^2\big), \ \forall\ x, y, z \in K. \tag{4.8}$$ Lips:cond

Moreover, in both [16, Algorithm 1] and [17, Algorithm 1], the authors needs the following relationship between the Lipschitz-type parameter η *and the modulus of strong quasiconvexity of the second argument of* f*,*

$$12\eta < \gamma, \tag{4.9}$$ rel:paramenters

which is not needed in Algorithm 1.

4.2 Numerical Experiments

The numerical behavior of our Algorithm 1 has been tested on the IMVI problem described in Example 4.1 with different sizes and inputs. The algorithms were implemented and executed in Python on a ASUS Laptop with Windows 11 and an AMD Ryzen 7 5800H CPU with 16GB RAM.

We tested our algorithm on problem (IMVI) with f_2 defined as in (4.7), that is,

$$f_2(x, y) := \langle x, F(y) - F(x)\rangle + G_2(y) - G_2(x),$$

$M_1 \leq \langle c, x\rangle + d \leq M_2$ for all $x \in K$, with $0 < M_1 < M_2 < +\infty$, F, G_2 defined as in Example 4.1, where $A, E_1, E_2 \in \mathbb{R}^{n\times n}$ be matrices, $b, c, g_1, g_2 \in \mathbb{R}^n$, $d, h_1, h_2 \in \mathbb{R}$ and E_1 and E_2 are positive definite matrices.

We present experiments with dimension $n = 1$, 2, 5, 10, 20 and 60. In the first two examples, we show the cases $n = 1, 2$.

Example 4.2. *($n = 1$) In the first experiment, we take* $K = [0, 5]$ *and*

$$A = 1, \quad E_1 = 2, \quad E_2 = 3, \quad b = 0, \quad c = 1,$$

$$g_1 = 1, \quad g_2 = 0, \quad h_1 = h_2 = d = 1.$$

This means that

$$F(x) = \frac{x}{x+1},$$

$$G(x) = \begin{cases} \frac{x^2+x+1}{x+1}, & \text{if } 0 \leq x \leq 2, \\ \frac{\frac{3}{2}x^2+1}{x+1}, & \text{if } 2 < x \leq 5, \end{cases}$$

and

$$f(x,y) = \begin{cases} \frac{y^2+y+xy+1}{y+1} - xF(x) - G(x), & \text{if } 0 \le y \le 2, \\ \frac{\frac{3}{2}y^2+xy+1}{y+1} - xF(x) - G(x), & \text{if } 2 < y \le 5. \end{cases}$$

Clearly, F is monotone on K and, for every $x \in K$, the function $f(x,\cdot)$ is strongly quasiconvex on K with modulus $\gamma = \frac{1}{3}$. Furthermore, f satisfies the Lipschitz-type condition (4.8) with $\eta = 1$, but condition (4.9) is violated, i.e., the algorithms in [16, Algorithm 1] and [17, Algorithm 1] can not be applied.

We test our algorithm with the parameters $\theta = 0.5$, $\epsilon = 0.9$, $\beta_k \equiv \frac{1}{4}$, $\alpha_k = \frac{10}{(k+1)^{\frac{3}{4}}}$, starting point $x_0 = 1$ and stopping criteria $\|x^k - y^k\| \le 10^{-3}$ or $\|x^k - x^{k+1}\| < 10^{-3}$. It terminates at the unique solution $x^* = 0$ after only 2 iterations.

Example 4.3. *($n = 2$) In the second experiment, we take $K = [0,5] \times [0,5]$,*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad d = 6,$$

$$E_1 = 2 \times I_2, \quad E_2 = 3 \times I_2,$$

$$g_1 = 1_2, \quad g_2 = 0_2, \quad h_1 = 1, \quad h_2 = 11.$$

For every $x \in K$, the function $f(x,\cdot)$ is strongly quasiconvex on K with modulus $\gamma = \frac{2}{11}$. We test our algorithm with the parameters $\theta = 0.5$, $\epsilon = 0.9$, $\beta_k \equiv \frac{1}{4}$, $\alpha_k = \frac{20}{(k+1)^{\frac{2}{3}}}$, starting point $x_0 = (1,1)$ and stopping criteria $\|x^k - y^k\| \le 10^{-3}$ or $\|x^k - x^{k+1}\| \le 10^{-3}$. We evaluate the performance of the algorithm by using the following error:

$$err = -\min_{y \in K} f(x^k, y).$$

Clearly, x^k is a solution of the equilibrium problem if and only if $err = 0$. It terminates after 18 iterations at $x^{18} = (0.46099256, 0)$ and

$$err = 1.0438606034668396e - 05.$$

Finally, we present the numerical experiments randomly generated in dimensions $n = 5, 10, 20$ and 50. We emphasize that in the next example, we compute 100 different problems for each dimension and Table 1 shows the average error and average CPU time of those 100 problems. In this experiment, we also compare the behavior of our algorithm (denoted by **StroExtra**) with the one in [39] (denoted by **Extra**) with the same choice of parameters.

ex4.4 **Example 4.4.** *In the last experiment, each entry of the vectors b, c, g_1, g_2 and scalars d, h_1, h_2 is randomly generated in the interval $(0,1)$. The matrices A, E_1, E_2 are also randomly generated such that each entries of these matrices are nonnegative and $E_1, E_2 \succeq I_n$. Then, the matrices E_1 and E_2 are scaled such that the strongly quasiconvex modulus of $f(x,\cdot)$ on $K = [0,5]^n$ is $\frac{1}{4}$.*

n		10	20	50
Error	**Extra**	0.06093646592072804	0.0041073317750288125	0.010153581387192747
	StroExtra	0.061555248932135556	2.5222626068245446e-07	3.935417598293578e-09
CPU time(s)	**Extra**	0.4870064544677735	0.8797023296356199	1.625372910499574
	StroExtra	0.3753505539894103	0.4021445512771605	0.18551418304443357

Table 1: Average errors and CPU time(s) of a 100 random problems in Example 4.4

table 1

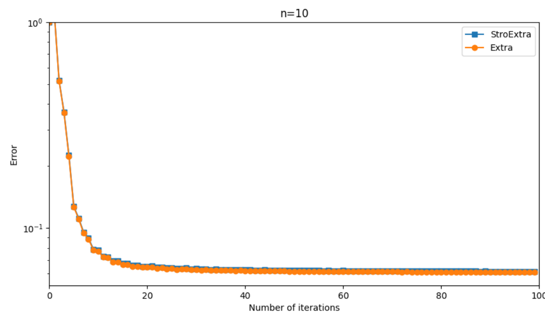
We use the parameters $\theta = 0.2$*,* $\epsilon = 0.9$*,* $\beta_k \equiv \frac{1}{4}$*,* $\alpha_k = \frac{8}{(k+1)^{\frac{2}{3}}}$*, starting point* $x_0 = (1, 1, \ldots, 1)$ *and stopping criteria* $\|x^k - y^k\| \leq 10^{-3}$ *or* $\|x^k - x^{k+1}\| < 10^{-3}$*. The performance of our algorithm* ***StroExtra*** *and* ***Extra*** *([39]) are evaluated by using the following error:*

$$err = \frac{\min_{y \in K} f(x^k, y)}{\min_{y \in K} f(x^0, y)}. \tag{4.10}$$

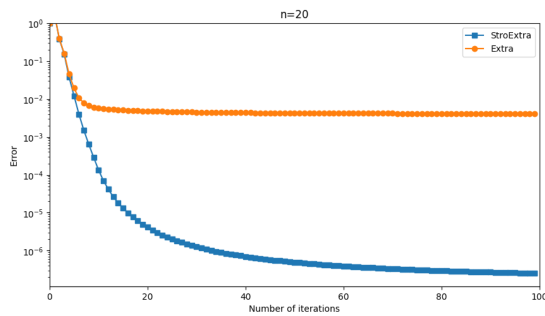
error:2

We test the algorithms for different values of n *and for each value, the average errors and CPU time(s) for* 100 *problems are reported in Table 1.*

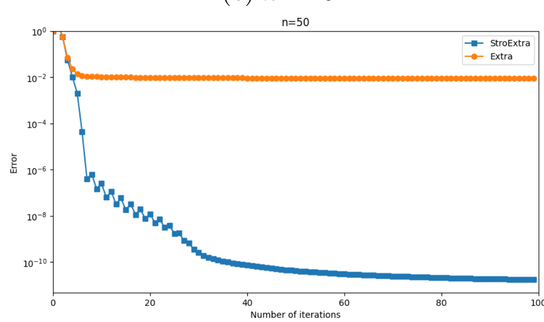
The behavior of the algorithms in the 100*-first iterations are also illustrated in Figure 1 for* $n = 10, 20, 50$*. For* $n = 10$*, two algorithms performs very similar. But for* $n = 20, 50$*, our algorithm achieves better solution in a reasonable time. Based on these experiments, we can say that our algorithm is competitive to the one in [39] and it can perform better in some special cases since the subgradient does not need to be normalized.*



(a) $n = 10$



(b) $n = 20$



(c) $n = 50$

fig 1

Figure 1: Average behavior of **StroExtra** and **Extra** for different values of n for a 100 random problems

5 Conclusions

We contributed to the discussion of proximal point type algorithms for classes of nonconvex equilibrium problems by providing, in this paper, an extragradient projection type method which uses a recently introduced generalized subdifferential for strongly quasiconvex functions. We proved that the sequence generated by this method converges to the unique solution of the problem without assuming a Lipschitz-type condition on the bifunction f and, moreover, a type of linear convergence rate was obtained.

We hope that this contribution could provide new lights in order to implement proximal point type algorithms for broader families of nonconvex functions under weaker assumptions.

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